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Symbolic Regression for the Wall Effect on a Settling Sphere in a Circular Cylinder

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Abstract

The main contribution of the current paper is to provide symbolic regression-based correlations for the wall correction factor of a sphere settling in a static fluid confined in an infinitely long circular cylinder. The first correlation will be valid for the entire spectrum of the flow regimes that range from the creeping flow regime up to the turbulent flow regime. While the second correlation is compact and only valid in the creeping flow regime, it accurately reassembles Haberman and Sayre's complicated analytical formula (David Taylor Model Basin Report No. 1143, Washington, D. C, US Navy Dept, 1958). A review of the available data and correlations is made to justify the selection of the data used in generating the correlation. This is followed by feeding a representative data to a symbolic regression software, selecting a dependency of the parameter of interest, running the software and selecting a correlation that has a relatively small length with good accuracy among the resulting suggested formulas. We used a small volume of training data from experiments to feed the symbolic regression machine-learning algorithm. The developed formula compared reasonably well with the available data and can extrapolate beyond its training data range. For the correlation limited for the creeping flow regime, the training data was obtained by solving a set of equations that constitute the Haberman and Sayre analytical formula. The obtained expression compared well when compared to the exact solution.

Keywords: Wall effect; Machine learning; Symbolic regression; Sphere settling in a circular cylinder.

INTRODUCTION

Analytical and experimental investigation of the motion of a moving sphere through otherwise quiescent fluid has been the subject of intensive research (Stokes 1851, Oseen 1910, Jones 1957, Satapathi 1960). Due to the sizable containers and tubes used, the experimental measurements are inevitably affected with the presence of a confining wall in many cases in which unbound fluid is assumed. When studying the drag on sphere in the presence of a confining wall it is found that the recirculatory wake formation is retarded along with the onset of separation (Clift, Grace et al. 1978). The fall velocity of a rigid sphere is retarded by the presence of confining walls as compared to the fall velocity at identical conditions in an unbound fluid (Chhabra 2006). The former observation is a consequence of the increase in the drag force felt by the falling sphere. Brown and Lawler (Brown and Lawler 2003) performed a detailed review of existing experimental data on the drag force on rigid spheres, excluding data with large uncertainty. They applied a wall correction factor to some of the data and were able to confirm the theoretical solution of Stokes drag in the creeping flow regime.

There are three different definitions of the wall correction factor (Clift, Grace et al. 1978) the least common one of them is the ratio of a hypothetical viscosity to the actual viscosity of the fluid. The hypothetical viscosity assumes Stokes flow and uses the observed terminal viscosity. The former definition is found convenient to use in falling ball viscometry (Clift, Grace et al. 1978, Singh, Sharma et al. 2012). The second definition is the ratio of the drag coefficient of a single settling sphere in bounded fluid to the drag coefficient in infinite medium at identical conditions. The third wall correction factor definition is the ratio of the terminal velocity at bounded fluid to the terminal velocity of the same particle at unbounded fluid. The three wall correction coefficients are the same at low Reynolds numbers (creeping flow)(Clift, Grace et al. 1978). As compared to the other two definitions, the velocity ratio is more extensively studied because it is a simpler and more convenient parameter to look at [12]. Most of the experimental literature is focused towards the velocity correction factor (e.g. (Fidleris and Whitmore 1961, Uhlherr and Chhabra 1995, Kehlenbeck and Felice 1999)). So the current paper considers seeking a correlation for the velocity correction factor. Theoretically derived formulas for wall correction factor at the creeping regime are only a function of the ratio of the diameter of the sphere to the diameter of the cylinder ($\lambda = d/D$).

In the creeping flow regime there exist a number of experimentally and theoretically developed formulas which vary significantly as the diameter ratio increases as indicated by Iwaoka and Ishii (IWAOKA and ISHII 1979). Nevertheless, the limit at which the viscous regime ends is not quite obvious (Haberman and Sayre 1958, Clift, Grace et al. 1978, Chhabra, Agarwal et al. 2003) and may not extend to measureable/practical values. It is accepted that there will be dependency on Reynolds number beyond the creeping flow regime (Oseen 1910). It worth noting that Reynolds number used in the formulas is based on the diameter of the sphere and the terminal velocity of the sphere at unbound fluid conditions rather than the existing terminal velocity when bounding wall exists. Fidleris and Whitmore (Fidleris and Whitmore 1961) and later Uhleherr and Chhabra (Uhlherr and Chhabra 1995) presented their correction factor in graphical format which is not practical to use. Based on computational experiments Wham et al. (Wham, Basaran et al. 1996) developed a drag correction factor formula that is applicable for $Re \leq 200$ and diameter ratios of $\lambda \leq 0.7$. The suggested model recovers the theoretical solution of Heaberman and Sayre as Re gets smaller. Chhabra et al. (Chhabra, Agarwal et al. 2003) rewrote the Wham's formula in terms of a velocity coefficient formula, but the resulting formulation is implicit and not practical to use. Di Felice (Di Felice, Gibilaro et al. 1995) and later Kehlenbeck and De Flice (Kehlenbeck and Felice 1999) proposed alternative empirical correlations for the intermediate regime, with the latter being more complex and more accurate (Chhabra, Agarwal et al. 2003).

At the fully turbulent regime many researchers suggest that the correction factor becomes again independent of Reynolds number. Uhleherr and Chbbrah suggest that the wall correction factor becomes independent of Re as Re exceeds 1000 (Uhlherr and Chhabra 1995). Similar to the creeping flow regime, Many formulas are derived for the high Reynolds number regime and they show notable differences (Uhlherr and Chhabra 1995). In fact some of the correlations are based on very limited data points (<10 data points) (Uhlherr and Chhabra 1995).

In general the wall correction factor is higher in the viscous regime than at higher Reynolds numbers. Chhabra et al. (Chhabra, Agarwal et al. 2003) suggests that "*It is generally agreed that the wall factor is independent of the Reynolds number both at very low and at very high values of the Reynolds number, while in between these two limiting behaviors, the wall factor is a function of both λ and Re in the intermediate transition regime*".

There are various correlations describing the wall effect, but they are scattered and inconsistent (IWAOKA and ISHII 1979, Chhabra, Agarwal et al. 2003). There is an opportunity to develop a new correlation using symbolic regression machine learning method. We propose that a generic function should depend on the diameter to wall ratio and the Reynolds number. This function can be developed in a similar way to the methods described in (Barati, Neyshabouri et al. 2014, El Hasadi and Padding 2019, El Hasadi and Padding 2023). Barati et al. (Barati, Neyshabouri et al. 2014) used symbolic regression to closely match almost all experimental data for the drag coefficient of a sphere for case of unbounded sphere. While, El Hasadi and Padding (El Hasadi and Padding 2023) used symbolic regression to investigate the underlying physics of the same problem. Li et al. (Li, Zhang et al. 2014), generated a generic formula of the wall correction factor for particles with different shapes settling in a cylindrical container filled with different medias (Newtonian and various non-Newtonian fluids) using artificial intelligence. The total number of experiments used to generate the generic formula was 513. Li et al. also worked on generating a wall correction factor of a sphere settling in a Newtonian fluid using a subset of their collected data. The generic model had only 55.5% of the data within 5% of uncertainty and 86% within 15% of uncertainty. The sphere in Newtonian fluid wall correction formula was found to have 99.4 % of the data within 5% of uncertainty. Although the sphere in a Newtonian fluid wall correction correlation is -thus- quite promising, no explicit formula is provided in the paper possibly because of the complicated form.

The main objectives of this paper are as follows:

- We aim to develop a generic correlation that depends on both λ and Reynolds number and covers all flow regimes. The selection of the data used to train the symbolic regression algorithm is described in the next section.
- We aim to find a simpler formula that accurately represents the exact solution of the problem in the viscous flow regime.

In the next section, we review the most common expressions in all flow regimes, with emphasis of available correlations in the intermediate regime. This is followed by a comparison of experimental data in the intermediate flow regime to each other, as well as to existing correlations. In the results section, we provide an evaluation of the performance of the generated correlation. The results section also includes a brief overview of the method we used to reproduce the exact solution of Haberman and Sayre, which we used to generate data for our symbolic regression machine learning algorithm.

MATERIALS AND METHODS

Recent reviews of the existing correlations that have been developed to estimate the drag coefficient can be found in (Chhabra, Agarwal et al. 2003, Arsenijević, Grbavčić et al. 2010, Singh, Sharma et al. 2012). Based on their extensive review of the bulk of the experimental work on the wall effect and its range of applicability, Chhabra et al. (Chhabra, Agarwal et al. 2003) recommended the use of the Haberman and Sayre equation (Haberman and Sayre 1958) in the viscous regime, the Di Felice equation (Di Felice 1996) for the intermediate regime, and the Newton equation (Barr 1931) for the turbulent regime. The aforementioned equations, which describe each flow regime, are listed below. This is followed by a review of all previous efforts to correlate the wall correction factor in the intermediate regime Haberman and Sayre tabulated values of their –theoretically derived- analytical solution for λ values ranging from 0 - 0.8 with an increment of 0.1. Moreover, they provided an approximate solution which takes the following form:

$$K_v = \frac{1-0.75857\lambda^5}{1-2.105\lambda+2.0865\lambda^3-1.7068\lambda^5+0.72603\lambda^6} \qquad (1)$$

Which is close enough to their exact analysis for up to λ =0.5. Numerical investigation by Bowen and Sharif (Bowen and Sharif 1994) found that the above formula to agree with numerical experiments data for up to λ =0.8 while Wham et al. (Wham, Basaran et al. 1996) found the agreement with the expression to extend for up to λ =0.5. Simulations performed by Tullock et al. (Tullock, Phan Thien et al. 1992) and Higdon and Muldowney (Higdon and Muldowney 1995) almost matched the exact analysis of Haberman and Sayre for up to λ=0.8.

Differences between Haberman and Sayre's expression with earlier analysis made by Ladenburg (Ladenburg 1907), and Faxen are almost identical. The superiority of HS solution comes from the wider applicability limits which extends much further than previous analysis (Haberman and Sayre 1958, Clift, Grace et al. 1978). The formula applicability limit also exceeds the theoretical analysis of Bohlin (Bohlin 1960) whom extended the original analysis of Faxen to formulate a more accurate expression for the wall correction factor (Bohlin 1960, Happel, Brenner et al. 1983).
Newton developed a semi-empirical formula (Barr 1931, Chandrasekhar 2003) for the fully turbulent flow regime can be expressed as:

$$K_v = (1-\lambda^2)(1-0.5\lambda^2)^{0.5} \qquad (2)$$

However, Bougas and Stamatoudis (Bougas and Stamatoudis 1993) investigation of the transient behavior of settling spheres at high Reynolds number found Munore's (Munroe 1889) formula to have better agreement as compared to other proposed correlations. Monroe's formula takes the form:

$$K_v = (1-\lambda)^{3/2} \qquad (3)$$

Bougas and Stamatoudis (Bougas and Stamatoudis 1993) applied Munore's (Munroe 1889) formula to their experimental work range (0.11 ≤ λ ≤ 0.83) and found a good agreement with the for values of λ up to 0.7 and Reynolds numbers ranging between 13500 - 70000.

Data and Correlations at Intermediate Regime

While many correlations exist in creeping flow and turbulent flow regimes, there is, however, a very limited attempts to find a correlation that covers the intermediate regime. Fidleres and Whitemore (Fidleris and Whitmore 1959, Fidleris and Whitmore 1961) pioneered the efforts to develop a correlation of the wall correction factor at intermediate regime as early as of late 50s of the last century but provided their results in graphical format. Their experimental data includes about 3000 experiments that covers a regime of Re ranging from 0.054 to 20000. Uhleherr and Chhabra (Uhlherr and Chhabra 1995) performed more than 220 experiments that covers almost the same Re range as of Fidleris and Whitmore. Their experimental results were summarized in a graphical format and was found to be systematically higher than Fidleris and Whitmore but they are within the experimental uncertainty. The deviation with other correlations ranged between good agreement and up to 70% difference with no clear trend of the discrepancy. De Flice (Di Felice 1996) provided a correlation that has Re dependency and later Kehlenbeck and Di Flice (Kehlenbeck and Felice 1999) improved the correlation of De Flice (Di Felice 1996) in which two fitting parameters are introduced and estimated from the available experimental data from the literature along with their own experimental measurements. The improved correlation includes the use of two fitting parameters and can be summarized as follows:

$$K_v = \frac{1-\lambda^p}{1-(\lambda/\lambda_o)^p} \quad (4)$$

Where the first fitting (λ_o) parameter is found to have the following Re dependency

$$\frac{\lambda_o - 0.283}{1.2-\lambda_o} = 0.041 Re^{0.524} \quad (5)$$

And the second fitting parameter (p) has the following Re dependencies

$$p = 1.44 + 0.5466 Re^{0.434}, Re \leq 35 \quad (6a)$$

$$p = 2.4 + 37.3 Re^{-0.8685}, Re \geq 35 \quad (6b)$$

Figure 1 presents the change of both fitting coefficients as Re changes (Eqs. (5 and 6)). Neither a physical interpretation of both coefficients was provided nor the significance of the assumed threshold of Re=35 was discussed in the paper. Although an alternative model in which an average value of p=2.2 was provided along with an optimized dependency of Re, the original model presented above agrees better with the experiments and will only be considered in the current work.

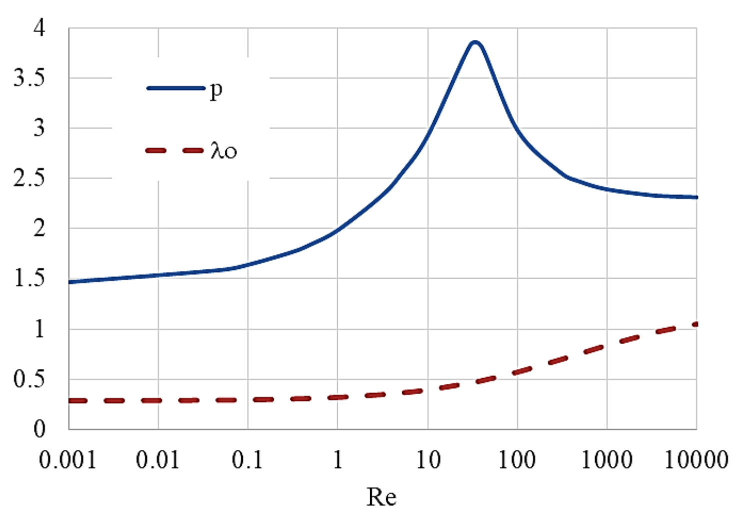

Figure (1). Graphical representation of fitting parameters used in Di Flice & Kehlenbeck correlation (Eqs. (5&6))

Figure 2 show the variation of the inverse of the velocity correction factor (Kv^{-1}) at different Re values. The solid lines are extracted from of Fidleris and Whitmore (FW) data along mapped with both Uhleherr and Chhabrah (UC) graphical results and Kehlenbeck and Di Flice (KD) correlation. The lines of Fidleris and Whitmore (solid) along Kehlenbeck and Di Flice correlation (dash-dot) with are –from top to bottom - at λ values of 0.05, 0.1, 0.2, 0.3, 0.4, 0.5 and 0.6 While those of Uhleherr and Chhabrah (dashed lines) are at λ values of 0.2-0.5 at an increment of 0.1. Both Haberman and Sayre "exact solution" (HS) and Newton's formula are shown as limiting cases as diamonds and triangles respectively. The creeping flow limit was assigned to a Reynolds number of 0.004 whereas the turbulent flow limit of Newton is assumed at a Reynolds number of 20000.

Interestingly the graphical representation from Uhleherr and Chhabra are only qualitatively similar to those of Fidleris and Whitmore. Although FW data is made at about the same time as the theoretical solution of Haberman and Sayre they show very good agreement at low Reynolds number. The agreement is also good at the fully turbulent regime limit when assuming Newton's formula to be applicable at Reynolds number of 20000. Fidleris and Whitmore (Fidleris and Whitmore 1961) stated that *"Newton's equation increases in reliability when the highest Reynolds numbers reached in the experiments, which were about 10000, are approached"*.

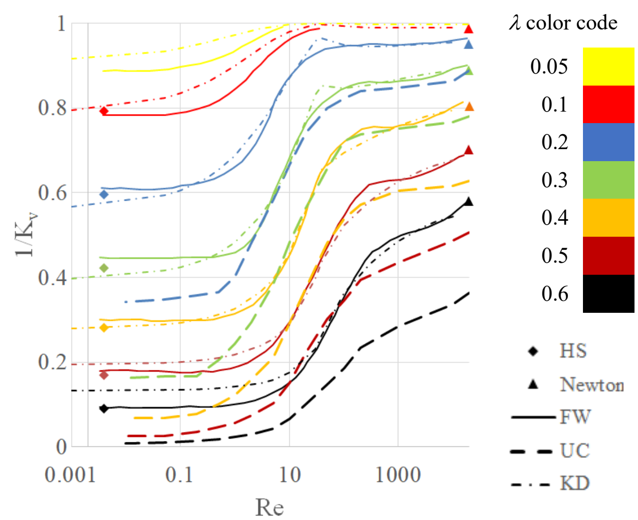

Figure (2). Graphical representation of data and correlations at intermediate regimes along with limiting formulas at creeping (Haberman & Sayre) and fully turbulent (Newton) regimes

Although the graphical representation of Fidleris and Whitmore show Re values of ~0.002 or so the smallest Re reported in the experiment is 0.054 so we suggest that F&W has extrapolated their plots a somewhat outside their experimental region. They assumed no further change with Reynolds number when extending their lines, however, by doing so there is a mismatch with the exact solution at most of the data points. On the other hand, Newton's formula agrees with the obtained trends at Re ~20000. If the trends would continue for another order of magnitude, chances are that even Newton's formula will lose its accuracy. In their investigation, Chhabra et al. (Chhabra, Agarwal et al. 2003) found the previously published data of (Uhlherr and Chhabra 1995) not to correlate well with the available empirical formulas at the intermediate flow regime. The justification was about the influence of non-verticality of the cylinder. Besides uncertainties that come from measurement of the different parameters, we suggest other experimental reasons such as sphericity, eccentricity between the sphere center and the cylinder center, estimating the terminal velocity would contribute to the error in the experimental data. If –for some reason- the sphere starts to rotate them the error would be exaggerated

Kehlenbeck and Di Felice (Kehlenbeck and Felice 1999) formula agrees with Newton's formula of the fully turbulent flow regime. However, the agreement with Haberman and Sayre is not guaranteed for all values of λ. This can be noticed at λ values of 0.6 where the prediction stalls at a higher value of ${K_v}^{-1}$ (thus under predicts the wall correction factor) and goes below the Haberman and Sayre solution of ${K_v}^{-1}$ (over predicting) for λ value of 0.3 and less. The hump at Re=35 at λ values of 0.3 and 0.2 are an artifact of the behavior of the parameter p as shown in Figure 1.

Care should be taken when considering the data to be fed to the software since some of the data may has higher uncertainties when the diameter ratio goes to 0.9 (Chhabra, Agarwal et al. 2003). The data from Uhleherr and Chhabra (Uhlherr and Chhabra 1995) should thus not be considered due to the possible high uncertainty involved in it. The experimental data of Fidleris and Whitemore (Fidleris and Whitmore 1961) is to be used by itself to develop the correlation. In spite of some mismatch with the exact solution of Heaberman and Sayre at some λ values, the correlation of Kehlenbeck and Di Felice (Kehlenbeck and Felice 1999) has reasonably good agreement with Fidleris and Whimore (Fidleris and Whitmore 1961) data. Generating a data from Kehlenbeck and Di Felice is not an option to avoid confusing the process by inclusion of different sources of error. Data from numerical simulations which does not cover a wide range of flow regime (e.g., [10, 11]) are also excluded.

Analysis software

The regression is to be carried-out using the Demo version of TuringBot symbolic regression software. The Demo version limits the number of data points to 50 and the number of columns to three (so that a variable can only be a function of other two parameters). The software lists a number of possible fitting functions (starting from a bare average) and provide their accuracy based on selected parameters (the default is the rms value of the target parameter). The software keeps refining –sometimes replacing- the existing formulas and producing more accurate functions until the user decides to stop it. The software allows for shuffling the data and selecting the Test/Train ratios. TuringBot interface allows for visualization of the different suggested solutions and how they fit the original data.

RESULTS

Here we present the regression formula obtained by the software and how it compares to the fed data and how it extrapolates outside the test/train regime. First, we seek a generic formula that properly describes the wall correction factor for viscous, inertia and extends to fully turbulent regime. Our goal is to find a functional dependency of the form $K_v=\phi(Re, \lambda)$. In the second section we will obtain a simple formula that is valid for the creeping flow regime using the training data from Haberman and Sayre's exact solution $HS=f(\lambda)$.

Generic Formula

In this part we seek a regression formula based on the graphical representation provided by Fidleris and Witmore (Fidleris and Whitmore 1961) and compare it with other available published data. Fidleris and Whitmore tabulated a small set of data (31 data points for Re of 0.1,

100, 1000, 3000, 10000) so we added some points from Kehlenbeck and De Flice at Re=10. Although the data is yet a small set we suggest that having the same systematic error –if any- would be more advantageous than using larger sets of different types of uncertainties. Fidleris and Whitemore provided their data for up to λ=0.6 thus there is a chance to judge correlations based on how they extrapolate when λ approaches 1 (${K_v}^{-1}$ should go to zero).
Among many formulas generated by the software we select the following Equation

$$ {K_v}^{-1} = (1-\lambda)^{\frac{1.3873}{0.7088+\emptyset}+1.1167\lambda} \qquad (7a)$$

where:

$$\emptyset = \left(-0.0542 + \frac{0.052}{\lambda}\right) \times (-0.2726 + Re)$$

is selected because simpler equations are not so accurate and more complicated functions are not encouraging for use because of their length and complexity. Eq. (7a) was obtained when all the data was used in the training with no data used in testing. This is because the sample of data point is limited especially when considering three parameters. Other functions were found to extrapolate poorly outside the training regime. As λ →1 (λ>0.95) Eq. (7a) has a singularity around Reynolds number of 1000 and was found to under-predict the correction factor at very low Reynolds numbers (Re → 0). Eq. (7a) has a good prediction for up to λ ~0.9 and may benefit from some refinement. By heuristic rounding of the coefficients of Eq. (7a) a modified version which gives almost the same prediction can be written as:

$$ {K_v}^{-1} = (1-\lambda)^{\frac{1.4042}{0.7071+f}+1.1167\lambda} \qquad (7b)$$

where

$$f = \left(-0.053 + \left(\frac{0.053}{\lambda}\right)\right) \times Re$$

Figure 3 provides a comparison between the graphical representations of FW data with the current predictions of the current correlations (Eqs. (7a & 7b)). The mismatch between the data which was used and the digitized plot of FW data is because Fidleris and Whitmore's data is averaged between two different plots of K_v and ${K_v}^{-1}$. We recommend not following the same strategy when looking at FW data but excluding the figure that show K_v since it uses a logarithmic axis and will be more sensitive to inaccuracies when extracting the data. In fact Brown and Lawler (Brown and Lawler 2003) followed the same recommended strategy when correcting the drag on sphere data. In addition to that the HS exact solution is plotted (assumed at Re=0.004) as solid diamonds, and Newton solution at Re=20000 as solid triangles. Both HS and Newton solution are shown at λ values of 0.1-0.6 (color code is the same as in Figure 2). As can be noted from Figure 3, a quite good agreement is obtained in the tested regime (λ up to 0.6). If there are any pitfalls, they would be that it misses Newton's formula at Re of 20000 at λ=0.6 and λ=0.5. The function fails to show the Re dependencies which are there is the experiment at Re>1000. Also, the exact solution at λ =0.1 is also missed at very low Re but agrees with the graphical presentation of Fidleris and Whitemore which is used in the training process. In spite of the very careful set up of the experiment and the low uncertainty reported by Fileris and Whitemore (Fidleris and Whitmore 1959) we suggest that there is an error involved in the experimental results at this λ value.

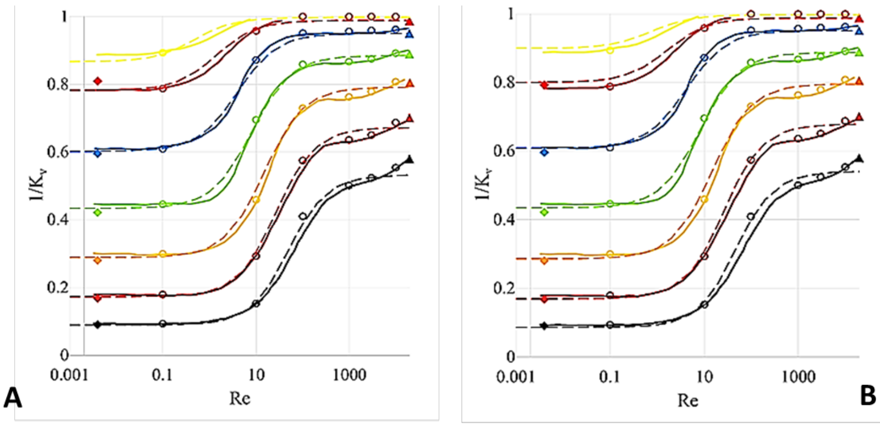

Figure (3). Agreement between the graphical representation of FW data along with limiting formulas at creeping (HS) and fully turbulent (Newton) regimes with (a) Eq. (7a) (b) Eq. (7b)

The only notable differences between Eqs. (7a) and (7b) are at low diameter ratios ($\lambda \leq 0.1$). In fact while the creeping flow limit of the inverse of the correction factor have been missed by both FW data and Eq. (7a), Eq. (7b) seem to be converging to the desired values as Re further reduced. The predictions of both Eqs. (7a) and (7b) are almost indistinguishable and only Eq. (7b) will be considered in further comparisons. Figure 4 provides a comparison of the predict-tions of Eq. (7b) at λ value outside the training regime (λ>0.6). Eqn. (7b) gives reasonably good agreement as it goes to zero. The data diverts from Newton's formula. The deviation from Newtons' formula is because the experimental data used in generating the formula seemingly departs from Newton's formula as λ extrapolates beyond λ=0.6. Further increase of Re beyond 10000 doesn't show a notable change in the predictions.

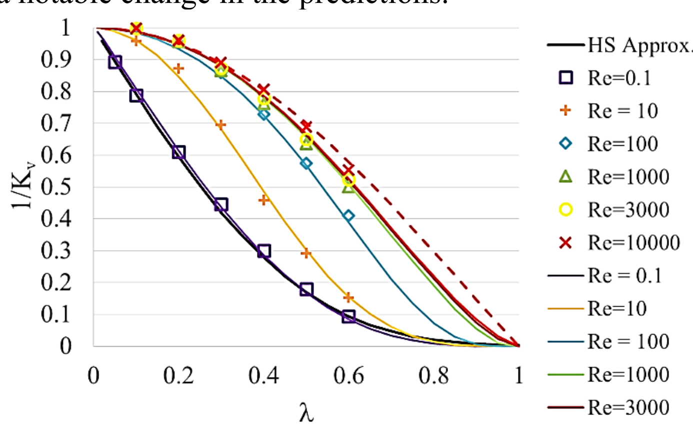

Figure (4). Predictions of Eq. (7b) at different diameter ratios.

So when excluding Reynolds numbers the only seen λ-terms that may appear in the exact solution–based on Eq. (7b) – would have the form of λ and (1- λ). At very low Reynolds numbers equation (7b) can be approximated as:

$$K_v^{-1} = (1-\lambda)^{2.147+1.1167\lambda} \qquad (8)$$

The other limiting behavior at very high Reynolds numbers is given by the current formula:

$$K_v^{-1} = (1-\lambda)^{1.1167\lambda} \quad (9)$$

A comparison of Eq. (8) with both Newton and Monrue's formulas (inverse of Eqs. 2 & 3 respectively) is provided in Figure 5. As can be seen Monure's predections are lower than predicted by Newton's formula since they might be based on experimental work that is possibly at higher Reynolds numbers than those of Newton. The current correlation is in better agreement with Newton's formula for up to λ=0.5 where it approaches Monrue's formula for up to λ=0.7. Beyond λ=0.8 both Newton and Monrue's predictions are about the same and both predicting a lower resistance by the wall than the current formula.

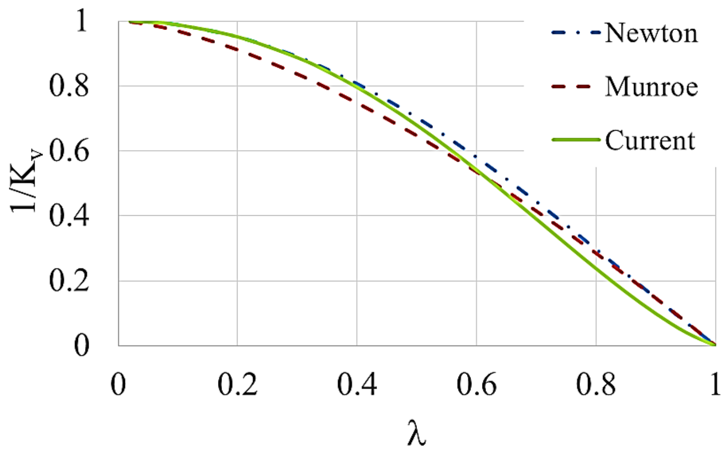

Figure (5). Comparison of limiting behavior of Eq. (9) with Newton and Monroe's formulas

Alternate expression for the Haberman and Sayre's exact solution

The suggested way of getting an alternative formula to the exact formula of Haberman and Sayre (Haberman and Sayre 1958) is to generate a large number of data from the exact solution itself. This means we should be able to solve the complicated exact solution, generate a large amount of data then feed it to the machine learning software to come-up with a simple formula.

Haberman and Sayre's (Haberman and Sayre 1958) solution is based on solving the Stokes stream function of a steady incompressible axisymmetric fluid in both cylindrical and spherical coordinates. Boundary conditions are applied to both solutions and the solution constants are relationship between the constants are obtained by direct comparison of terms. A linear algebraic system of equation is then constructed to obtain the constants. For the motion of a rigid sphere in a stationary liquid, the resulting system takes the form:

$$\begin{aligned}&\frac{(2n-1)(n!)}{4(2n-3)}\frac{b_{n-2}}{R^{n-1}}\pi+\frac{n(n-1)(n!)}{4}\frac{b_n}{R^{n+1}}\pi\\&+\sum_{m=0,2,\ldots}^{\infty}\left[\frac{1}{(n-2)!}S_3^{n+m-1}+\frac{n(n-1)}{(n-2)!\,(2n-3)}S_4^{n+m-2}\right]\frac{b_m}{R^{m+1}}\lambda^{n+m-1}\\&+\frac{(2n+1)(n!)}{4}\frac{a_{n-1}}{R^{n+1}}\pi\\&+\sum_{m=1,3,\ldots}^{\infty}\left[\frac{-1}{(n-2)!}S_4^{n+m-1}+\frac{1}{(n-4)!\,(2n-3)}S_2^{n+m-2}\right]\frac{a_m}{R^{m+2}}\lambda^{n+m}=\alpha\end{aligned} \quad (10a)$$

Where R is the radius of the cylinder and the parameters S are integrals of combinations of

$$\frac{n!}{4}\frac{b_{n-2}}{R^{n-1}}\pi+\frac{(2n-1)n(n-1)(n!)}{4(2n+1)}\frac{b_n}{R^{n+1}}\pi-\sum_{m=0,2,\dots}^{\infty}\left[\frac{S_4^{n+m}}{(n-2)!\,(2n+1)}\right]\frac{b_m}{R^{m+1}}\lambda^{n+m-1}$$
$$+\frac{(2n-1)(n!)}{4}\frac{a_{n-1}}{R^{n+1}}\pi-\sum_{m=1,3,\dots}^{\infty}\left[\frac{S_2^{n+m}}{(n-2)!\,(2n+1)}\right]\frac{a_m}{R^{m+2}}\lambda^{n+m+2}=0 \quad (10b)$$

modified Bessel functions of zeroth, first and second order. The reader is referred to the original report for detailed analysis and description of the solution technique. The constants a_n and b_n constitute the solution vector and can be solved to a higher degree of accuracy by increasing the number of solved equations *"n"*. Since two equations are solved simultaneously, n takes an even number (typically multiples of 2).
The parameter α in Eq. (7a) can be defined as follows:

$$\alpha=\begin{cases}-U & n=2\\ 0 & n>2\end{cases} \quad (11)$$

In fact, only b_0 is of interest in solving the wall correction factor. It can be shown that [17]:

$$Kv=\frac{-2}{3}\frac{b_0}{UR} \quad (12)$$

Similar to many exact solutions the analysis yielded some infinite series and the selection of number of terms (truncation) would affect the accuracy of the solution. The process of determination of $\boldsymbol{b_o}$ is described by Haberman and Sayre as:
"Wall correction factors for rigid spheres moving in a still liquid inside an infinitely long cylinder have been determined by numerically solving the algebraic system (Equation [10]) for the coefficient $\boldsymbol{b_o}$ *over a large range of diameter ratios. The number of equations of the algebraic system used was increased (at most up to eight) until only very small changes in the value of* $\boldsymbol{b_o}$ *were obtained".*

A computer program was developed to numerically compute the integrals S_2, S_4 and S_3 and construct the system of linear equations. The linear system is then solved using the buit-in Octave function (Octave 4.4.1 version is used) which produced the solution. The computed values of b_0 are then used to calculate the wall correction factor as defined by Eq. (9). Table (1) lists a comparison of the reported values of the exact and approximate solution along with the values of the obtained results at different system sizes. The 2x2 system in the above equation should recover the approximate solution which –physically- represents a situation where a slip boundary condition is imposed at the cylinder. The length of the system used to obtain the exact solution is 8x8 as per Haberman and Sayre reported.

"The number of equations of the algebraic system used was increased (at most up to eight) until only very small changes in the value of b_o were Obtained."

A comparison of the results of the different systems is provided in Table (1) below

The second columns in Table (1) is borrowed from Haberman and Sayre report while the third column is generated using Eq. (1). Typically, there should be no difference between the third, fourth columns as well as the second and sixth columns and the existing differences are because of numerical differences in evaluating the integrals of S (although not shown these differences are not significant- ranging between 0.01% -0.5%) and because of how truncation error grow when solving the linear system. The built-in linear system solver is designed to find the efficient method of solving a linear system and perform necessary operations such as scaling. In fact a 16x16 system had poor results at λ value > 0.5, this is because the values of S grows dramatically as *n* increases affecting the condition number of the matrix to values that could not be fixed by the linear solver. The solver is capable of getting a solution at lower λ values is because the integrals S is multiplied by powers of λ than alleviate this issue at small λ values. The difference between the solution of the 8x8 system is comparable to the exact solution with a maximum discrepancy of 1.5% at λ=0.8.

Table (1). Comparison of the numerical solution at different system sizes with HS solution

λ	Exact HS	App. (Eq. 1)	2x2	4x4	8x8	%diff with Exact
0	1.000	1	1	1	1	0
0.1	1.236	1.2633	1.2632	1.2632	1.2632	0.02
0.2	1.680	1.6797	1.6794	1.6795	1.6795	-0.03
0.3	2.371	2.3697	2.3687	2.3700	2.3701	-0.04
0.4	3.596	3.5816	3.579	3.5913	3.5914	0.63
0.5	5.970	5.8700	5.8596	5.9467	5.9474	-0.38
0.6	11.135	10.593	10.55	11.082	11.092	-0.39
0.7	24.955	21.425	21.215	24.519	24.676	-1.12
0.8	73.555	49.023	47.711	71.387	74.648	1.49
0.9	NA	121.27	111.83	323.74	460.95	NA

41 data points were fed to the software ranging between 0-0.8 with a uniform increment of 0.02. Complicated formulas are excluded since we are seeking a simple expression. By assuming $Kv=f(\lambda,(1-\lambda))$ and using test/train ratio of 50-50, the following function is selected:

$$K_v = (1-\lambda)^{-2.68945} - (\lambda + 0.4967)\lambda \quad (13a)$$

Which can be further simplified as:

$$K_v = (1-\lambda)^{-2.69} - (\lambda + 0.5)\lambda \quad (13b)$$

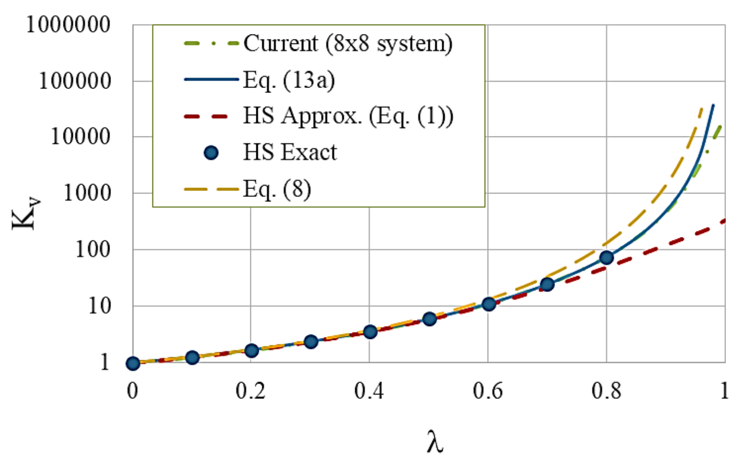

Figure (5). Comparison with Eq. (13a) along with HS exact and approximate analysis

Differences between Eqs. 13a and 13b are graphically indistinguishable. The agreement with the report exact solution from Haberman and Sayre's report along with the agreement with the 8x8 system result is listed in the Table (2).

Although a difference of ~ 6% can be seen at λ=0.9, we suggest that the behavior of the formula is acceptable and could be more realistic than the theoretical analysis which –to some extent- has numerical fingerprints. The formula has a singularity at λ =1 which is physically indicates that the sphere would cease its motion for infinitely long time if the sphere diameter is the same as the cylinder diameter. Such phenomena can be noticed in the falling ball viscometer for viscous liquids when λ approaches one. Indeed, the current expression agrees better with the exact solution reported by Haberman and Sayre than the limiting behavior of the generic function produced earlier (Eq. (8)).

Table (2). Comparison of current expressions with data fed to the software and HS exact solution

λ	Exact HS	8×8	Eqn. 13a	Diff %		Eqn. 13b	Diff %	
				HS	8×8		HS	8×8
0	1	1	1	0	0	1	0	0
0.1	1.263	1.2632	1.2679	0.4	0.4	1.2677	0.4	0.4
0.2	1.68	1.6795	1.6830	0.2	0.2	1.6826	0.2	0.2
0.3	2.371	2.3701	2.3708	-0.0	0.0	2.3703	-0.0	0.0
0.4	3.596	3.5914	3.5918	0.6	0.0	3.5916	0.6	0.0
0.5	5.97	5.9474	5.9523	-0.3	0.1	5.9531	-0.3	0.1
0.6	11.135	11.092	11.098	-0.3	0.1	11.101	-0.3	0.1
0.7	24.955	24.676	24.646	-1.2	-0.1	24.660	-1.2	-0.1
0.8	73.555	74.648	74.794	1.7	0.2	74.858	1.8	0.3
0.9	--	460.95	487.91	--	5.9	488.52	--	6.0
1	--	22098	NA	--	NA	NA	--	NA

CONCLUSIONS

A machine-learning-based symbolic regression was used to generate a generic wall correction formula. The formula, based on data reported by Fidleris and Whitmore, captures the general trend of the data and has good agreement with intermediate and limiting behavior in both creeping flow and fully turbulent regimes.

The generated formula (Eq. (7a)) was found applicable for λ values between 0.02 and 0.9 and Reynolds numbers up to 20000 whereas a slightly modified version (Eq. (7b)) is assumed applicable for almost the complete range of λ ($0 \leq \lambda \leq 1$) for the same Reynolds number. The exact solution of Haberman and Sayre was used to create a simple formula for the creeping flow regime, which was compared with the exact solution and found to be accurate up to λ =0.8. However, more data is needed at very high Reynolds numbers to confirm that the effect of Re diminishes as it approaches zero. Future work could include using more data from FW or running benchmark numerical simulations to improve the accuracy of the regression formula.

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