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Modification of Logistic growth model

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Abstract

The population problem has been a world concern since the British economist Malthus (1798) proposed his exponential model, ever since many researches have been carried out to justify the model. Most successful predictive models are shown to be based on extended forms of the classical Verhulst logistic growth equation

$$\frac{dn}{dt} = rn(1-n/k) \tag{1}$$

Starting from this simple ordinary differential equation, several models have been chosen exquisitely in this study until access finally the generalized Fisher equation

$$u_t = u_{xx} + u(1-u^n) \tag{2}$$

Exp-Function Method used to solve this partial differential equation. For a particular value $n=1$ and by the help of symbolic computation of Maple, the obtained results show that Fisher equation is a good approximation for a single species dynamic.

Key Words: *Logistic Model, Gompertz growth equation, Fisher Equation, Exp-Function Method.*

الملخص

إن مسألة نمو المجتمع استحوذت على اهتمام العالم منذ أن اقترح الاقتصادي الإنجليزي مالثوس سنة 8971 نموذجه الأسي، وهناك عدد هائل من الأبحاث قدمت لتحسين وضبط هذا النموذج وكان أنجحها توقعا تلك التي تأسست على النموذج اللوجستي وتعديلاته.

$$\frac{dn}{dt} = rn(1-\frac{n}{k})$$

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انطلاقا من هذه المعادلة التفاضلية البسيطة، تم اختيار عدة نماذج في هذه الدراسة للوصول إلى معادلة فيشر العامة.

$$u_x = u_{xx} + u(1 - u^n)$$

ولحل هذه المعادلة التفاضلية الجزئية استخدمت طريقة الدالة الأسية من أجل قيمة خاصة n=1 وبالاستفادة من برنامج Maple فإن النتائج المتحصل عليها بينت أن معادلة فيشر تعتبر تقريبا جيدا لوصف ديناميكية الكائن المفرد.

الكلمات المفتاحية: نموذج لوجستي، معادلة نمو جومبيرتز، معادلة فيشر، طريقة المعادلة التفاضلة الجزئية.

1. Introduction

1.1 Exponential Growth Model

Malthus suggested that the single species grows according to the following ordinary differential equation

$$\frac{dn}{dt} = f(n) = rn(t) \tag{3}$$

where n is the population size at time t and $r(>0)$, the population growth rate.

Integrating and putting $n = n_0$ at $t = t_0$, yields the instantaneous population size as

$$n(t) = n_0 \exp(rt) \tag{4}$$

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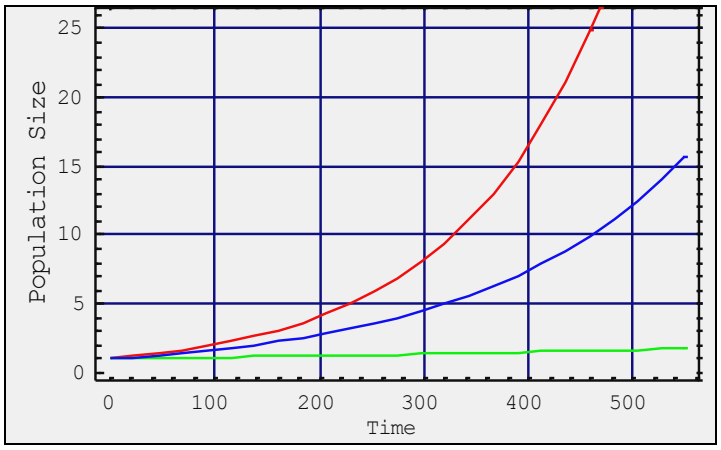


Figure 1 Exponential Growth; *r*=0.001,0.005,0.007

1.2 Logistic Growth Model

Because the approximation obtained with Malthus model was less than desirable, since the population cannot grow exponentially for all time, without taking account of external conditions, such as limitation of space, food supply, or other resources, so another model is needed. The Logistic (or Verhulst) equation is

$$\frac{dn}{dt} = rn(1-n/k)$$

where $k(\neq 0)$, is the carrying capacity of the habitat.

Integrating and putting $n = n_0$ at $t = t_0$, gives

$$n(t) = n_0 k/(n_0 + (k-n_0)\exp(-rt)). \qquad (5)$$

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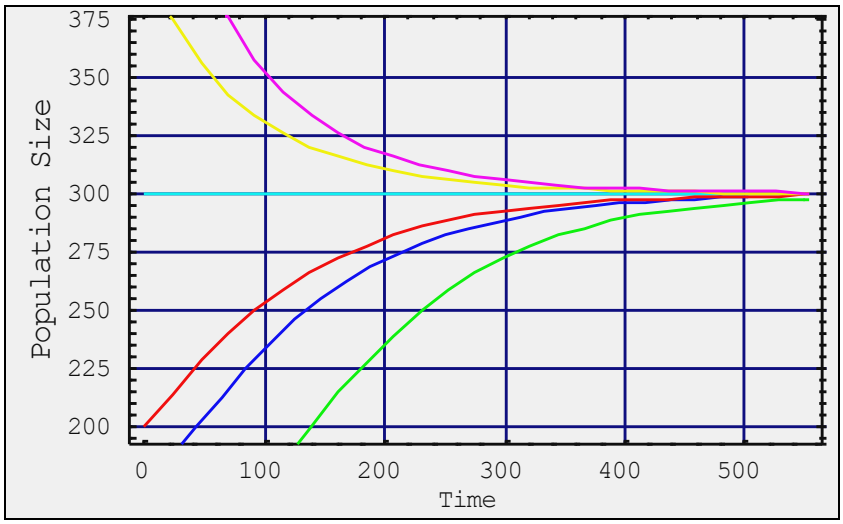


Figure 2 Logistic Growth; *k*=300, $n_0 = 100,130,200,400,500$.

2. Extended logistic model

Since the original work, Eq. (1), there have been several contributions suggesting alternative functions $f(n)$. Here are some examples [9].

2.1 Generic Growth function

Turner and co-authors [10] proposed a model as modification of logistic equation, which has a form

$$\frac{dn}{dt} = rn^{1+\beta(1-\gamma)}\left[1-\left(\frac{n}{k}\right)^{\beta}\right]^{\gamma} \tag{6}$$

where β, γ are positive exponents and $\gamma < 1 + \frac{1}{\beta}$. This has the solution,

$$n(t) = k/\left[1+\left[(\gamma-1)\beta r k^{\beta(1-\gamma)}t + \left[\left(\frac{k}{n_0}\right)^{\beta}-1\right]^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right]^{\frac{1}{\beta}} \tag{7}$$

2.2 Blumberg's equation

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$$\frac{dn}{dt} = rn^{\propto}\left[1-\frac{n}{k}\right]^{r}. \tag{8}$$

The population at the inflection point is given by [1]

$$n_{inf} = \frac{\propto}{\propto+\gamma}k \tag{9}$$

2.3 Von Bertalanffy's growth equation

Von Bertalanffy [11] proposed the following form which is a special case of the Bernoulli differential equation

$$\frac{dn}{dt} = rn^{\frac{2}{3}}\left[1-\left(\frac{n}{k}\right)^{\frac{1}{3}}\right] \tag{10}$$

which has solution

$$n(t) = k\left[1+\left[1-\left(\frac{n_0}{k}\right)^{\frac{1}{3}}\right]e^{-\frac{1}{3}rk^{-\frac{1}{3}}t}\right]^3 \tag{11}$$

2.4 Richards growth equation

He extended Bertalanffy's growth equation by the following model which is also a special case of the Bernoulli differential equation[6]

$$\frac{dn}{dt} = rn\left[1-\left(\frac{n}{k}\right)^{\beta}\right] \tag{12}$$

which has solution

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$$n(t) = k\left[1 - e^{-\beta rt}\left[1 - \left(\frac{n_0}{k}\right)^{-\beta}\right]\right]^{-\frac{1}{\beta}} \tag{13}$$

2.5 Gompertz growth equation

The growth rate modeled by the Gompertz function is given by [9]

$$\frac{dn}{dt} = rn\left[ln\left(\frac{k}{n}\right)\right]^{\gamma} \tag{14}$$

which has solution [10]

$$n(t) = k\,exp\left\{\left[\ln\left(\frac{n_0}{k}\right)\right]^{1-\gamma} + r'(-1)^{\gamma}(1-\gamma)t\right\}^{\frac{1}{1-\gamma}} \tag{15}$$

2.6 Generalized logistic growth function

Tsoularis [9] defined the following Generalized logistic growth equation which incorporates all previously reported forms as special cases

$$\frac{dn}{dt} = rn^{\alpha}\left[1 - \left(\frac{n}{k}\right)^{\beta}\right]^{\gamma} \tag{16}$$

The population at the inflection point is given by

$$n_{inf} = \left(1 + \frac{\beta\gamma}{\alpha}\right)^{-\frac{1}{\beta}}k \tag{17}$$

These were some treatments to develop the logistic model, such treatments however should not restrict on ordinary differential equations, since it will be more realistic if $f(n)$ in equation (3) becomes a function of more than one variable as

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observed in the real world. To fill this gap and for other purposes, Fisher proposed nonlinear partial differential equation (18) below.

3. Fisher Equation

Fisher's equation was one-dimensional and had a specific logistic reaction term [3]

$$u_t = Du_{xx} + ru(1 - \frac{n}{k})$$

Fisher proposed this equation as a model of diffusion of a species in a 1D habitat; *D* is the diffusion constant, *r* is the growth rate of the species, and *K* is the carrying capacity. A dimensionless version of the equation takes the form

$$u_t = u_{xx} + u(1 - u) \tag{18}$$

It has wide applications in the fields of logistic population growth, flame propagation, neurophysiology, autocatalytic chemical reactions, and nuclear reactor theory [2], [8].

3.1 Solutions of nonlinear PDEs

Many methods have been proposed for solving the nonlinear differential equations, in recent years, see[4], [5], [7] and the introductions in [2], [10] where many methods have been listed. One of the effective methods is Exp-Function Method, Emine and Yusuf [2] applied this method to solve the generalized Fisher equation with higher order non-linearity

$$u_t = u_{xx} + u(1 - u^n) \tag{19}$$

Their study could summarized as follows

Define the complex variation μ as $\mu = kx + wt$, equation (19) becomes

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$$k^2u'' - wu' - u^{n+1} + u = 0 \tag{20}$$

where *k*, *w* are real parameters and the prime denotes the derivative with respect to μ.

Making the transformation

$$u = v^{\frac{1}{n}} \tag{21}$$

equation (20) converts into ordinary differential equation

$$k^2nvv'' + k^2(1-n)(v')^2 - wnvv' - n^2v^3 + n^2v^2 = 0 \tag{22}$$

Assuming the solution of equation (22) can be expressed in the form

$$v(\mu) = \frac{\sum_{n=-c}^{d} a_n \exp(n\mu)}{\sum_{m=-p}^{q} b_m \exp(m\mu)} \tag{23}$$

where *c*, *d*, *p* and *q* are positive integers, a_n and b_m are unknown constants. For simplicity, set $p = c = 1$and $d = q = 1$, then equation (23) reduces to

$$v(\mu) = \frac{a_1 \exp(\mu) + a_0 + a_{-1}\exp(-\mu)}{\exp(\mu) + b_0 + b_{-1}\exp(-\mu)} \tag{24}$$

Substituting Eq. (24) into Eq. (22), equating to zero the coefficients of all powers of $\exp(n\mu)$, $(n = -4, -3, \dots, 3,4)$ yields a set of algebraic equations for $a_{-1}, a_0, a_1, b_1, b_0, b_1, k, w$. By using symbolic computation of Matlab, Mathematica or so on, solving this system, and after some mathematical manipulations they obtained two cases [2]

case 1:

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$$u(x,t)=\left[\frac{\cosh\left(\mp A(x\mp Bt)\right)+\sinh\left(\mp A(x\mp Bt)\right)}{\mp 2+\cosh\left(\mp A(x\mp Bt)\right)}\right]^{\frac{1}{n}}, \tag{25}$$

where $A=\frac{n}{\sqrt{2(n+2)}}$ and $B=\frac{n+4}{\sqrt{2(n+2)}}$.

case 2:

$$u(x,t)=\left[\frac{\cosh\left(\mp A(x\mp Bt)\right)-\sinh\left(\mp A(x\mp Bt)\right)}{\mp 2+\cosh\left(\mp A(x\mp Bt)\right)}\right]^{\frac{1}{n}}, \tag{26}$$

where $A=\frac{n}{\sqrt{2(n+2)}}$ and $B=\frac{n+4}{\sqrt{2(n+2)}}$.

3.2 Special Case of the Generalized Fisher equation

In the previous sub-section Exp-Function Method for solving the Generalized Fisher equation (19) has introduced, hence Eq. (18) is a special case of Eq. (19) that is when $n=1$, putting this value of n in equations (25), (26) gives the solutions

$$u(x,t)=\left[\frac{\cosh\left(\mp A(x\mp Bt)\right)+\sinh\left(\mp A(x\mp Bt)\right)}{\mp 2+\cosh\left(\mp A(x\mp Bt)\right)}\right], \tag{27}$$

and

$$u(x,t)=\left[\frac{\cosh\left(\mp A(x\mp Bt)\right)-\sinh\left(\mp A(x\mp Bt)\right)}{\mp 2+\cosh\left(\mp A(x\mp Bt)\right)}\right], \tag{28}$$

where $A=\frac{1}{\sqrt{6}}$ and $B=\frac{5}{\sqrt{6}}$.

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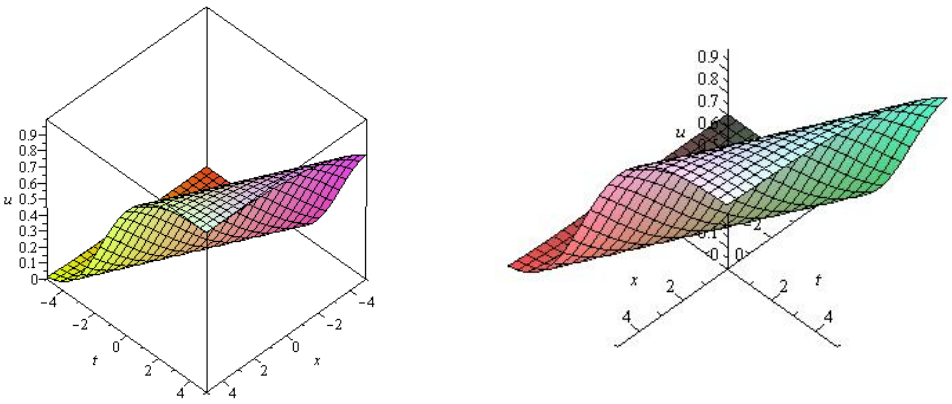


Figure 3 Fisher equation; $n = 1$

4. Discussion

Malthus model and hence logistic growth model have been used as a basis for several extended models to develop the classical model. The majority of models were some kinds of ordinary differential equations. Such treatments however should not restrict on ordinary differential equations, since it will be more realistic if $f(n)$ in equation (3) becomes a function of more than one variable as observed in the real life. To fill this gap, Fisher non-linear partial differential equation is introduced and its solution discussed using Exp-Function Method which is one of the latest and effective methods. The obtained results show that Fisher equation is a good modification to logistic model.

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