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Order Continuous Operators

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Abstract

The order continuous operators consider one of important topic in functional analysis and its applications, the affiliations among order continuous operators and the other classes of operators such as σ-order are continuous, order bounded, and singular operators, have been studied and investigated, we proved that if an order bounded operator $\mathcal{F}: \mathcal{U} \to \mathcal{V}$ concerning two Riesz space with $\mathcal{V}$ Dedekind complete is continuous and ordered, then $|\mathcal{F}|$ is order continuous, and this paper shows that if $\mathcal{U}$ is space that is countable, now $\mathcal{F}$ is not σ-order continuous, while $\mathcal{U}$ is uncountable, then $\mathcal{F}$ is necessarily σ-order continuous, by giving an example we showed that null ideal for the operator $\mathcal{F}$ is band when $\mathcal{F}$ is bounded ordered, further, it is ordered and continuous. Finally, we concluded the operator that is a positively and orderly continuous map on ordered dense with memorizing Riesz subspace of a Riesz space with its range is Dedekind complete, it has only unique ordered continuous expansion all of space.

Keywords: Riesz Spaces, Positive Operator, Order Continuous Operator, Order Bounded Operator.

INTRODUCTION

In linear algebra and functional analysis, the operator $\mathcal{F}: \mathcal{U} \to \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are Riesz spaces is defined as an order continuous operator whenever $u_\alpha \overset{0}{\to} 0$ in $\mathcal{U}$ gives $\mathcal{F}(u_\alpha) \overset{0}{\to} 0$ inside $\mathcal{V}$; and σ-order continuous whenever $u_n \overset{0}{\to} 0$ in $\mathcal{U}$ implies $\mathcal{F}(u_n) \overset{0}{\to} 0$ in $\mathcal{V}$. Many researchers studied this topic, and some of these previous studies have proven the theory of extending the positive order continuous operators(Veksler, 1960). Another study defined new classes of operators, which are so-called unbounded order continuous and further boundedly unbounded order continuous operators and gave extra settings under which uo-continuity is equivalent to order continuity of some operators on Riesz spaces (Bahramnezhad & Azar, 2018). The report investigated the relationships located between order to topology continuous operators and different types of operators for example b-weakly compact, order weakly compact and order continuous operators, and studied adjoint of order to norm continuous operators (Jalili et al., 2021). While recent study extends the properties of unbounded order continuous operators from $\mathcal{U}$ Riesz space into $\mathfrak{R}$ (Turan et al., 2022). Aydin and Gorokhova studied the concept of statistically continuous and bounded operators with statistically ordered convergent sequences on Riesz spaces (Aydin & Statistics, 2023). In this paper, some basic results from the theory of order continuous operators were studied with proofs as needed. First, this work introduces some of its types: σ-order continuous operators, order bounded operators, singular operators, and order continuous components of the positive operator, in addition, it shows some of their properties as well as the relations between them, then it studied of describing the group of all continuously ordered operators and some their useful characterizations and some of its application.

Concepts and Theories:

Definition 1 from previous study (Abramovich & Aliprantis, 2002):
$\mathcal{F}: \mathcal{U} \to \mathcal{V}$ is called positive if $\mathcal{F}(u) \geq 0 \; \forall \, u \in \mathcal{U} \, ; u \geq 0$.
The beginning point within the theory of positive administrators may be a principal expansion theorem of Kantorovic (Kantorovitch, 1940), who proved that additive operator $\mathcal{F}: \mathcal{U}^+ \to \mathcal{V}^+$to be the constraint of a single positive operator from $\mathcal{U}$ into $\mathcal{V}$. The details follow.

Theorem 1 by Kantorovic (Kantorovitch, 1940)
If $\mathcal{F}: \mathcal{U}^+ \to \mathcal{V}^+$ is an additive operator, then $\mathcal{F}$ has a uniquely positive extension to the entire space $\mathcal{U}$, the unique extension (denoted by $\mathcal{F}$ again) is given by

$$\mathcal{F}(u) = \mathcal{F}(u^+) - \mathcal{F}(u^-) \qquad \forall \; u \in X \; .$$

Definition 2 from previous study (Kreyszig, 1978):
The *disjoint complement* V^d is defined as follows:

$V^d = \{ u \in \mathcal{U} : u \perp v \;\; \forall v \in V \}$ Where V^{dd} means $(V^d)^d$.

Definition 3 from previous study (Aliprantis & Burkinshaw, 1985)*:*
If $\mathcal{U}$ Riesz space and $U \subset \mathcal{U}$, then U is named *solid* if $\forall \, u \in U, v \in \mathcal{U}$ and $|u| \leq |v|$ gives $v \in U$.
If U is solid sub space from $\mathcal{U}$, then U is named Ideal of space $\mathcal{U}$.

Definition 4 John (John, 1990)*:*
The ordered closed ideal is mentioned to be a *band.*

Definition 5 Aliprntis (Aliprantis & Burkinshaw, 1985):
If $\mathcal{B}$ is a band in Riesz space $\mathcal{U}$ and the following condition $\mathcal{U} = \mathcal{B} \oplus \mathcal{B}^d$ is met, then $\mathcal{B}$ is called a *projection band.*
The band does not have to be a projection band; we need to mention the following theorem which was proven by Riesz (Riesz, 2000).

Theorem 2 Aliprntis (Aliprantis & Burkinshaw, 1985):
If $\mathcal{B}$ is a vector which is a band in a Dedekind of complete Riesz space $\mathcal{U}$, formerly $\mathcal{U} = \mathcal{B} \oplus \mathcal{B}^d$ satisfied.

Definition 6 Ogasawara (Ogasawara, 1942). A net $\{ u_\alpha \}$ in a Riesz space is called the order convergent to u whenever there exists a net $\{ v_\alpha \}$ with the same indexed set satisfying $|u_\alpha - u| \leq v_\alpha \downarrow 0$ it is referred to as: $u_\alpha \overset{0}{\to} u$

Definition 7 Ogasawara (Ogasawara, 1942)*:* The operator $\mathcal{F}: \mathcal{U} \longrightarrow \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are two Riesz spaces is named to be
(a) Ordered continuous, if $u_\alpha \overset{0}{\to} 0$ inside $\mathcal{U}$ then $\mathcal{F}(u_\alpha) \overset{0}{\to} 0$ in $\mathcal{V}$.
(b) δ- ordered continuous, if $u_n \overset{0}{\to} 0$ inside $\mathcal{U}$ then $\mathcal{F}(u_{n)} \overset{0}{\to} 0$ in $\mathcal{V}$.
It is valuable to remember that a positive operator $\mathcal{F}: \mathcal{U} \to \mathcal{V}$ is ordered continuous $\Leftrightarrow$ $u_\alpha \downarrow 0$ in $\mathcal{U}$ gives $\mathcal{F}(u_\alpha) \downarrow 0$ in $\mathcal{V}$ (moreover $\Leftrightarrow$ $0 \leq u_\alpha \uparrow u$ in $\mathcal{U}$ gives $\mathcal{F}(u_\alpha) \uparrow \mathcal{F}(\mathrm{u})$ in $\mathcal{V}$).
the next example proves that δ- order continuous does not have to be an ordered continuous operator.

Example 1 Let $\mathcal{U}$ the vector space of all real functions that are Lebesgue integrable on the period $0 \leq u \leq 1$. It is clear that any two functions that are not equal at one point are considered two unequal functions, meaning of that $\mathcal{P} \geq \mathcal{Q}$ for all $0 \leq u \leq 1$, this is called the law of pointwise ordering of functions, $\mathcal{U}$ describes a Riesz space (for more details a function space). Further, see that $\mathcal{P}_\alpha \uparrow \mathcal{P}$ applied in $\mathcal{U}$ $\Leftrightarrow$ $\mathcal{P}_\alpha(u) \uparrow \mathcal{P}(u)$ satisfied of all $0 \leq u \leq 1$ in $\mathfrak{R}$.
If the operator $\mathcal{F}: \mathcal{U} \to \mathcal{R}$ is defined as follows

$$\mathcal{F}(\mathcal{P}) = \int_0^1 \mathcal{P}(u)du$$

It is clear that $\mathcal{F}$ is not an order continuous operator that is also positive, hence from the Lebesgue Dominated Convergence theory it is clear that $\mathcal{F}$ is δ- order continuous.
The following theorem shows that order continuous operators that are order bounded have a sum of useful characterizations.

Theorem 3 If $\mathcal{U}, \mathcal{V}$ are Riesz spaces where Y is Dedekind complete, and $\mathcal{F}: \mathcal{U} \longrightarrow \mathcal{V}$ is bounded ordered operator, then following arguments are equivalent:

a) The operator $\mathcal{F}$ is continuous.
b) If $u_\alpha \downarrow 0$ gives in $\mathcal{U}$, then $\mathcal{F}(u_\alpha) \overset{0}{\to} 0$ in $\mathcal{V}$.
c) If $u_\alpha \downarrow 0$ gives in $\mathcal{U}$, then $\inf\{|\mathcal{F}(u_\alpha)|\} = 0$ in $\mathcal{V}$.
d) $\mathcal{F}^+$ and $\mathcal{F}^-$are together order continuous.
e) $|\mathcal{F}|$ is ordered continuous.

Proof: a) → b) and b) → c) are clear.
c) → d) to prove this paragraph, it is sufficient to prove that $\mathcal{F}^+$ is order continuous, suppose $u_\alpha \downarrow 0$ in $\mathcal{U}$, and $\mathcal{F}^+(u_\alpha) \downarrow s \geq 0$ in $\mathcal{V}$. It is need to prove that $s = 0$, Make some β fixed and let $u = u_\beta$.
Now $\forall\, v \in [o, u]$ and $\forall\, \alpha \geq \beta$ is:

$$0 \leq v - v \wedge u_\alpha = v \wedge u - v \wedge u_\alpha \leq u - u_\alpha$$

This means that:

$$\mathcal{F}(v) - \mathcal{F}(v - u_\alpha) = \mathcal{F}(v - v \wedge u_\alpha) \leq \mathcal{F}^+(u - u_\alpha) = \mathcal{F}^+(u) - \mathcal{F}^+(u_\alpha)$$ Therefore

$$0 \leq s \leq \mathcal{F}^+(u_\alpha) \leq \mathcal{F}^+(u) + |\mathcal{F}(v \wedge u_\alpha)| - \mathcal{F}(v) \qquad \dots\dots\dots (\star)$$

applies $\forall\ \alpha \geq \beta$ and $\forall\ 0 \leq v \leq u$, since $\forall\ 0 \leq v \leq u$ gives $v \wedge u_\alpha \downarrow_{\alpha \geq \beta} 0$ this means that it fulfills the hypothesis $inf_{\alpha \geq \beta}\{|\mathcal{F}(v \wedge u_\alpha)|\} = 0$, referring to the inequality $(\star)$ note that $0 \leq s \leq \mathcal{F}^+(u) - \mathcal{F}(v)$ satisfies $\forall\ 0 \leq v \leq u$, from $\mathcal{F}^+(u) = \sup\{\mathcal{F}(v): 0 \leq v \leq u\}$, recent inequality leads to that $s = 0$,this is what is required,
d) → e) since $|\mathcal{F}| = \mathcal{F}^+ + \mathcal{F}^-$, this means that $|\mathcal{F}|$ is order continuous,
e) → a) this can be easily proven by applying the inequality $|\mathcal{F}(u)| \leq |\mathcal{F}|(|u|)$.
theorem 3 is true for σ-order continuous operators, and this can be proven in the same way as before.
The combination of all ordered continuous operators of $\ell_b(\mathcal{U}, \mathcal{V})$ is represented by the symbol $\ell_n(\mathcal{U}, \mathcal{V})$.
The symbol $\ell_n(\mathcal{U}, v)$ indicates that the order continuous operators are in addition normal operators, in another meaning:

$$\ell_n(\mathcal{U}, \mathcal{V}) = \{\mathcal{F} \in \ell_b(\mathcal{U}, \mathcal{V}); \mathcal{F}\ is\ order\ continuous\}.$$

The symbol $\ell_c(\mathcal{U}, \mathcal{V})$ symbolizes the collection of all ordered bounded operators from $\mathcal{U}$ into $\mathcal{V}$ which are σ-ordered continuous, in another meaning:

Obviously, $\ell_n(\mathcal{U}, \mathcal{V})$, $\ell_c(\mathcal{U}, v)$ $\ell_C(\mathcal{U}, \mathcal{V}) = \{\mathcal{F} \in \ell_b(\mathcal{U}, \mathcal{V}); \mathcal{F}\ is\ \sigma\text{–}\ order\ continuous\}$ are both two vectors subspaces of $\ell_b(\mathcal{U}, \mathcal{V})$ furthermore $\ell_n(\mathcal{U}, \mathcal{V}) \subseteq \ell_c(\mathcal{U}, \mathcal{V})$ satisfied. Once $\mathcal{V}$ is Dedekind complete. T. Ogasawara [14] proved that together $\ell_n(\mathcal{U}, \mathcal{V})$, $\ell_c(\mathcal{U}, \mathcal{V})$ are seen as bands of $\ell_b(\mathcal{U}, \mathcal{V})$, The next theorems and details follow.

Theorem 4 Ogasawara (Ogasawara, 1942): If $\mathcal{U}, \mathcal{V}$ are two Riesz spaces where $\mathcal{V}$ is Dedekind complete, this means that $\ell_n(\mathcal{U}, \mathcal{V})$, $\ell_c(\mathcal{U}, \mathcal{V})$ are together bands of $\ell_b(\mathcal{U}, \mathcal{V})$.

Proof: It shall create that $\ell_n(\mathcal{U}, \mathcal{V})$ is a band of $\ell_b(\mathcal{U}, \mathcal{V})$. That $\ell_c(\mathcal{U}, \mathcal{V})$ is a band can be demonstrated in a comparable way.

See firstly that if $|E| \leq |\mathcal{F}|$ satiefies in $\ell_b(\mathcal{U}, \mathcal{V})$ with $\mathcal{F} \in \ell_n(\mathcal{U}, \mathcal{V})$, therefore from Theorem 3 gives that $E \in \ell_n(\mathcal{U}, \mathcal{V})$, which is, $\ell_n(\mathcal{U}, \mathcal{V})$ is an ideal of $\ell_b(\mathcal{U}, \mathcal{V})$.

To prove that $\ell_n(\mathcal{U}, \mathcal{V})$ is a band, let $0 \leq \mathcal{F}_\lambda(u) \uparrow \mathcal{F}(u)$ inside $\ell_b(\mathcal{U}, \mathcal{V})$ with $\{\mathcal{F}_\lambda\} \subseteq \ell_n(\mathcal{U}, \mathcal{V})$, and assume that $0 \leq u_\alpha \uparrow u$ in $\mathcal{U}$ At that time for λ fixed. We have

$$0 \leq \mathcal{F}(u - u_\alpha) \leq \big(\mathcal{F}(u) - \mathcal{F}_\lambda(u)\big) + \mathcal{F}_\lambda(u - u_\alpha)$$

Since $u - u_\alpha \downarrow 0$ means that

$$0 \leq \inf\{\mathcal{F}(u - u_\alpha)\} \leq (\mathcal{F}(u) - \mathcal{F}_\lambda(u)) \ \forall \lambda.$$

This shows that $\mathcal{F} - \mathcal{F}_\alpha \downarrow 0 \implies \inf(\mathcal{F}(u - u_\alpha)) = 0$ and then $\mathcal{F}(u_\alpha) \uparrow \mathcal{F}(u)$, In conclusion $\mathcal{F} \in \ell_n(\mathcal{U}, \mathcal{V})$.

The symbol $\ell_s(\mathcal{U}, \mathcal{V})$ denotes the ensemble of all order bounded operators from Riesz space $\mathcal{U}$into Riesz space $\mathcal{V}$, where $\mathcal{V}$ is Dedekind complete, which are disjoint from $\ell_c(\mathcal{U}, \mathcal{V})$, in another meaning $\ell_s(\mathcal{U}, \mathcal{V}) = \ell_c^d(\mathcal{U}, \mathcal{V})$

and its nonzero elements will be mentioned to as *singular operators*. Because $\ell_b(\mathcal{U}, \mathcal{V})$ is a Dedekind complete Riesz space, this is given as a following from Theorem 2 that $\ell_c(\mathcal{U}, \mathcal{V})$ is a projection band, this proves that $\ell_b(\mathcal{U}, \mathcal{V}) = \ell_c(\mathcal{U}, \mathcal{V}) \oplus \ell_s(\mathcal{U}, \mathcal{V})$.

Especially, if $\mathcal{F}$ is order bounded operator from Riesz space $\mathcal{U}$ into Riesz space $\mathcal{V}$, then $\mathcal{F}$ has a unique decomposition $\mathcal{F} = \mathcal{F}_c + \mathcal{F}_s$; $\mathcal{F}_c \in \ell_c(\mathcal{U}, \mathcal{V})$, $\mathcal{F}_s \in \ell_s(\mathcal{U}, \mathcal{V})$. The operator $\mathcal{F}_s$ is known as singular component, and $\mathcal{F}_c$ is known as the σ-order continuous component of $\mathcal{F}$.

It may happen that $\ell_c(\mathcal{U}, \mathcal{V}) = \{0\}$.

Another basic imbalance is valuable in numerous considers and was presented by T. Ando, which is later called as *Ando s inequality.*

If $\mathcal{U}$ a Riesz space, $u, v \in \mathcal{U}$, and γ is a real number, then from identity $u - v = (1-\gamma)u + (\gamma u - v)$ we find that

$$u - v \leq (1-\gamma)u + (\gamma u - v)^+$$

The σ-order continuous the following basic imbalance is valuable in numerous thinks about and was presented by

Theorem 5 supposes that $\mathcal{U}$ *and* $\mathcal{V}$ are two Riesz spaces where $\mathcal{V}$ is Dedekind complete, and $\mathcal{F}: \mathcal{U} \longrightarrow \mathcal{V}$ is a positive operator, the following sentences are true $\forall\, u \in \mathcal{U}\,; u \geq 0$

1. $\mathcal{F}_c(u) = \inf\{\sup \mathcal{F}(u_n): 0 \leq u_n \uparrow u\}$, and
2. $\mathcal{F}_n(u) = \inf\{\sup \mathcal{F}(u_\alpha): 0 \leq u_\alpha \uparrow u\}$.

Proof: Frist, let's demonstrate the formula for $\mathcal{F}_n$

For each positive operator $L: \mathcal{U} \longrightarrow \mathcal{V}$ define $L^\circ: \mathcal{U}^+ \longrightarrow \mathcal{V}^+$ by

$$L^\circ(u) = \inf\{\sup L(u_\alpha): 0 \leq u_\alpha \uparrow u\} \ \forall\, u \in \mathcal{U}^+.$$

Obviously, $0 \leq L^\circ(u) \leq L(u) \ \forall\, u \in \mathcal{U}^+$, and $L^\circ(u) = L(u)$ whenever $L \in \ell_n(\mathcal{U}, \mathcal{V})$.

Furthermore, it is easy to see that L° is improver on $\mathcal{U}^+$, and then (use Theorem 1), covers to a positive operator from $\mathcal{U}$ into $\mathcal{V}$. On the other hand, it is easy to see that $L \longrightarrow L^\circ$ from $\ell_b^+(\mathcal{U}, \mathcal{V})$ into $\ell_b^+(\mathcal{U}, \mathcal{V})$, is likewise additive, $(L_1 + L_2)^\circ = L_1^\circ + L_2^\circ$ satisfies, thus $L \longrightarrow L^\circ$ defines a positive Operator Form $\ell_b(\mathcal{U}, \mathcal{V})$ into $\ell_b(\mathcal{U}, \mathcal{V})$. From the inequality $0 \leq L^\circ \leq L$ can be seen that $L \longrightarrow L^\circ$ is ordered continuous, $L_\alpha \downarrow 0$implies $L_\alpha^\circ \downarrow 0$.

Now let $\mathcal{F}: \mathcal{U} \longrightarrow \mathcal{V}$ be positive operator that is fixed. It is sufficient to demonstrat that $\mathcal{F}^\circ$ is ordered continuous. When this is proven, then $\mathcal{F}^\circ \leq \mathcal{F}$ implies $\mathcal{F}^\circ = (\mathcal{F}^\circ)_n \leq \mathcal{F}_n$, and since $\mathcal{F}_n \leq \mathcal{F}^\circ$ holds trivially, we see that $\mathcal{F}^\circ = \mathcal{F}_n$. Finally, let $0 \leq v_\lambda \uparrow v$ in $\mathcal{V}$ it should be shown that $\mathcal{F}^\circ(v - v_\lambda) \downarrow 0$ in $\mathcal{V}$.

Fix the interval $\varepsilon \in (0, 1)$, and suppose $\mathcal{F}_\lambda$ denote the operator defined by:

$$\mathcal{F}_A(u) = \sup\{\mathcal{F}(v): 0 \leq v \leq u, v \in A, u \in \mathcal{U}\} \ A \text{ ideal of } \mathcal{U}$$

that decides with $\mathcal{F}$ on the ideal created by $(\varepsilon v - v_\lambda)^+$, disappears on $(\varepsilon v - v_\lambda)^-$. Evidently, $\mathcal{F} \geq \mathcal{F}_\lambda \downarrow \geq 0$ and $\mathcal{F}_\lambda(v_\lambda - \varepsilon v)^+ = 0$ satisfied $\forall \lambda$.
Let $\mathcal{F}_\lambda \downarrow E$ in $\ell_b(\mathcal{U}, \mathcal{V})$. Since $R(v_\lambda - \varepsilon v)^+ = 0 \ \ \forall \lambda$ and $0 \leq (v_\lambda - \varepsilon v)^+ \uparrow (1-\varepsilon)v$, this means $E^\circ(v) = 0$.
From the Ando inequality

$$0 \leq (v - v_\lambda) \leq (1-\varepsilon)v + (\varepsilon v - v_\lambda)^+ \text{ , it follows that}$$

$$0 \leq \mathcal{F}^\circ(v - v_\lambda) \leq (1-\varepsilon)\mathcal{F}^\circ(v) + \mathcal{F}^\circ(\varepsilon v - v_\lambda)^+ \quad \text{........ (1)}$$

Now due to the reason $0 \leq u \leq (\varepsilon v - v_\lambda)^+$ gives $\mathcal{F}_\lambda(u) = \mathcal{F}(u)$, the following gives:

$$\mathcal{F}^\circ((\varepsilon v - v_\lambda)^+) = inf\,\{sup\,\mathcal{F}\,(u_\alpha): 0 \leq u_\alpha \uparrow (\varepsilon v - v_\lambda)^+\}$$
$$= inf\,\{sup\,\mathcal{F}_\lambda\,(u_\alpha): 0 \leq u_\alpha \uparrow (\varepsilon v - v_\lambda)^+\}$$
$$= \mathcal{F}_\lambda^\circ(\varepsilon v - v_\lambda)^+ \leq \mathcal{F}_\lambda^\circ(v)$$

By substituting into the equation (1) the following is obtained:

$$0 \leq \mathcal{F}^\circ(v - v_\lambda) \leq (1-\varepsilon)\mathcal{F}^\circ(v) + \mathcal{F}_\lambda^\circ(v) \quad \text{......... (2)}$$

Since $L \longrightarrow L^\circ$ is order continuous and $\mathcal{F}_\lambda \downarrow E$, this means that $\mathcal{F}_\lambda^\circ \downarrow E^\circ$, in other words, $\mathcal{F}_\lambda^\circ(v) \downarrow E^\circ(v) = 0$ applying inequality (2) we get

$$0 \leq \inf\{\mathcal{F}^\circ(v - v_\lambda)\} \leq (1-\varepsilon)\,\mathcal{F}^\circ(v)$$

satisfied for all $0 < \varepsilon < 1$, therefore $\mathcal{F}^\circ(v - v_\lambda) \downarrow 0$, as desired.
Let us assume that the order bounded operator $\mathcal{F}: \mathcal{U} \longrightarrow \mathcal{V}$ is between the two Riesz spaces $\mathcal{U}, \mathcal{V}$ where $\mathcal{V}$ is Dedekind completed. Then define the *null ideal* by $N_\mathcal{F}$ of $\mathcal{F}$ as follows:

$$N_\mathcal{F} = \{u \in \mathcal{U}: |\mathcal{F}|(|u|) = 0\}$$

It can be seen that $N_\mathcal{F}$ is ideal of $\mathcal{U}$.
The *carrier* of $\mathcal{F}$ is defined as the disjoint complement of $N_\mathcal{F}$, which is denoted by the symbol $C_\mathcal{F}$, meaning that:

$$C_\mathcal{F} = N_\mathcal{F}^d\{u \in \mathcal{U}: u \perp N_\mathcal{F}\}$$

obviously, $|\mathcal{F}|$ is positive on $C_\mathcal{F}$, $0 < u < C_\mathcal{F} \Leftrightarrow 0 < |\mathcal{F}|(u)$.
The following example shows that if the operator is order bounded and is order continuous, then its null ideal is a band, but the opposite is not true.

Example 2 assume that $\mathcal{U}$ is an infinite set, and that $\mathcal{U}_\infty = \mathcal{U} \cup \{\infty\}$ is the One-point compactification of $\mathcal{U}$ considered with the Discrete Topology, Therefore, a function $g: \mathcal{U} \longrightarrow \mathcal{R}$ belongs to $C(\mathcal{U}_\infty) \Leftrightarrow$ there is at least one constant a (depending on g) so that $\forall\ \varepsilon > 0$, there is $|g(h) - a| < \varepsilon\ \forall$ values of u except a limited number of u, this mean that $g(\infty) = a$.
Now let's make a fixed countable subset $\{u_1, u_2, ...\}$ of $\mathcal{U}$, and then define the operator $\mathcal{F}: C(\mathcal{U}_\infty) \longrightarrow \mathcal{R}$ by

$$\mathcal{F}(g) = g(\infty) + \textstyle\sum_1^\infty 2^{-n} g(u_n)$$

It is clear that $\mathcal{F}$ is a positive operator, also

$$N_\mathcal{F} = \{g \in C(\mathcal{U}_\infty): g(u_n) = 0, \forall\, n = 1,2, ...\}$$

because $g_\alpha \uparrow g$ satisfies in $C(\mathcal{U}_\infty) \Leftrightarrow g_\alpha(u) \uparrow g(u)$ satisfies in $\mathcal{R}\ \ \forall\, u \in \mathcal{U}$, it gives that $N_\mathcal{F}$ is a band of $C(\mathcal{U}_\infty)$, however, we assertion that $\mathcal{F}$ is not order continuous.
To make sure in this, suppose the net $\{u_\alpha\} \subset C(\mathcal{U}_\infty)$ with α goes over the collection of all finite subsets of $\mathcal{U}$ Therefore, $0 \leq u_\alpha \uparrow \mathbf{1}$ satisfies in $C(\mathcal{U}_\infty)$ Although $\mathcal{F}(u_\alpha)$ not implies $\mathcal{F}(\mathbf{1})$.
Moreover, it is motivating to see that if $\mathcal{U}$ countable, thus $\mathcal{F}$ is not σ-order continuous, whereas if $\mathcal{U}$ is uncountable, then $\mathcal{F}$ must be σ-order continuous.
ideal A is said to be ***σ-ideal*** if the following is true $\{u_n\} \subseteq A, 0 \leq u_n \uparrow u \Rightarrow u \in A$.

Theorem 6: If $\mathcal{F}: \mathcal{U} \longrightarrow \mathcal{V}$ is order bounded operator between two Riesz spaces with $\mathcal{V}$ Dedekind complete, and $A_\mathcal{F}$ is the ideal generated by $\mathcal{F}$ in $\ell_b(\mathcal{U}, \mathcal{V})$, then $\mathcal{F}$ achieves the following:

1. the null ideal N_L is a band for any $L \in A_\mathcal{F} \Leftrightarrow \mathcal{F}$ is order continuous.
2. the null ideal N_L is a σ-ideal for any operator $L \in A_\mathcal{F} \Leftrightarrow \mathcal{F}$ is σ-order continuous.

Proof: To begin it's important only to prove (1) because the proof of (2) is similar. The "only if" part follows directly from Theorem 3 for the "if" part (in view of Theorem 3) we can suppose that $\mathcal{F} \geq 0$, let $0 \leq u_\alpha \uparrow u$ in $\mathcal{U}$ and $0 \leq \mathcal{F}(u_\alpha) \uparrow v \leq \mathcal{F}(u)$ in $\mathcal{V}$. The next to be shown $\mathcal{F}(u) = v$.
By the end, assume $0 < \varepsilon < 1\ \forall \alpha$, let $\mathcal{F}_\alpha$ be the operator given by relation (1) in Theorem 2 that approves with $\mathcal{F}$ on the ideal created by $(\varepsilon u - u_\alpha)^+$and vanishes on $(\varepsilon u - u_\alpha)^-$, it is clear that: $\mathcal{F} \geq \mathcal{F}_\alpha \uparrow \geq 0$,and $\mathcal{F}_\alpha(\varepsilon u - u_\alpha)^-$ for every α, let $\mathcal{F}_\alpha \downarrow L \geq 0$ in $\ell_b(\mathcal{U}, \mathcal{V})$ it is clear that $L \in A_{\mathcal{F}}$, and $L(\varepsilon u - u_\alpha)^- = 0$ satisfies $\forall\ \alpha$ and so $\{(\varepsilon u - u_\alpha)^-\} \subseteq N_L$ However, $0 \leq (\varepsilon u - u_\alpha)^- \uparrow (1-\varepsilon)u$ in $\mathcal{U}$ and thus, because by our hypothesis N_L is a band, $u \in N_L$ In conclusion, $L(u) = 0$
Finally, the relation:

$$0 \leq \mathcal{F}(\varepsilon u - u_\alpha)^+ = \mathcal{F}_\alpha(\varepsilon u - u)^+ \leq \mathcal{F}_\alpha(u)$$

combined upon Ando is inequality $0 \leq u - u_\alpha \leq (1-\varepsilon)u + (\varepsilon u - u_\alpha)^+$ implies $0 \leq \mathcal{F}(u) - v \leq \mathcal{F}(u - u_\alpha) \leq (1-\varepsilon)\mathcal{F}(u) + \mathcal{F}(\varepsilon u - u_\alpha)^+ \leq (1-\varepsilon)\mathcal{F}(u) + \mathcal{F}_\alpha(u)$
Considering that $\mathcal{F}_\alpha(u) \downarrow L(u) = 0$ the last inequality yields $0 \leq \mathcal{F}(u) - v \leq (1-\varepsilon)\,\mathcal{F}(u)$ $\forall\ \varepsilon\ \in (0,1)$ Hence, $\mathcal{F}(u) = v$ holds.
To clarify the previous theorem, suppose the operator $\mathcal{F}: C(H_\infty) \to R$ is defined as in the previous example by

$$\mathcal{F}(g) = g(\infty) + \sum_{1}^{\infty} 2^{-n} g(h_n)$$

As it has been seen before,
$N_F = \{g \in C(H_\infty): g(h_n) = 0\ \forall\ n = 1,2,\ldots\}$, and this means that N_F is a band of $C(H_\infty)$. On the other hand, if $L: C(H_\infty) \to R$ is defined by

$$L(g) = g(\infty)$$

Then L is a positive operator satisfying $0 \leq L \leq \mathcal{F}$, it is clear that $N_L = \{g \in C(H_\infty): g(\infty) = 0\}$, it is clear that the net $\{h_\alpha\}$ of all characteristic functions of the finite subsets of *H* satisfies $\{h_n\} \subset N_L$ and $h_\alpha \uparrow \mathbf{1}$, as $\mathbf{1} \notin N_L$, not that N_L is not a band of $C(H_\infty)$, in accordance with Theorem 6 (1), let $\mathcal{U}$ and $\mathcal{V}$ be two Riesz spaces with $\mathcal{V}$ Dedekind complete, if $C_{\mathcal{F}} = \{0\}$, then $\mathcal{F} \in \ell_b(\mathcal{U}, \mathcal{V})$ is said to have *zero carrier*. It is clear to see that the zero administrators are arranged as persistent administrators with zero carriers. On the other hand, if $C_{\mathcal{F}} = \{0\}$, then $\mathcal{F} \perp \ell_b(\mathcal{U}, \mathcal{V})$. (To see this, write $\mathcal{F} = \mathcal{F}_n + \mathcal{F}_\sigma$, and note that $|\mathcal{F}| = |\mathcal{F}_n| + |\mathcal{F}_\sigma|$ So $N_{\mathcal{F}} \subseteq N_{\mathcal{F}_n}$, hence depending on the order denseness of $N_{\mathcal{F}}$ note that $N_{\mathcal{F}_n} = \mathcal{U}$, $\mathcal{F}_n = 0$ and so $\mathcal{F} = \mathcal{F}_\sigma \in \ell_b(\mathcal{U}, \mathcal{V})$, From $|\mathcal{F} + L| \leq |\mathcal{F}| + |L|$, it follows that $N_{\mathcal{F}} \cap N_L \subseteq N_{L+\mathcal{F}}$, and utilizing the truth that the crossing point of two arrange thick beliefs is an arrange thick perfect, note that the administrators of $\ell_b(\mathcal{U}, \mathcal{V})$ with zero carriers shape a perfect.
Another hypothesis tells us that this perfect is continuously arranged thick in $\ell_b(\mathcal{U}, \mathcal{V})$.

Theorem 7: Let $\mathcal{U}$ and $\mathcal{V}$ be two Riesz spaces with $\mathcal{V}$ Dedekind complete. Then the ideal $\mathcal{F} \in \ell_\sigma(\mathcal{U}, \mathcal{V}): C_r = \{0\}$ is order dense in $\ell_\sigma(\mathcal{U}, \mathcal{V})$.

Proof: since the set $\mathcal{F} \in \ell_\sigma(\mathcal{U}, \mathcal{V}): C_r = \{0\}$ is an ideal in $\ell_\sigma(\mathcal{U}, \mathcal{V})$, let $\mathcal{F} \in \ell_\sigma(\mathcal{U}, \mathcal{V})$ is the positive operator with zero carrier.
Since $\mathcal{F}$ is not order continuous, there exists (use Theorem 6) an operator $0 \leq L \leq \mathcal{F}$ where N_L is not a band, denote by B the band created by N_L, then let R be the operator firm by Theorem 6 where $R = L$ on $R = 0$ and مراجعة on B^d , it is clear that $N_L \subseteq N_R$, and $0 \leq R \leq L$. On the other hand, later $R = 0$ holds on $C_L = N_L^d = B^d$, we see that $N_L \oplus C_L \subseteq N_R$, and this (in the opinion of Theorem 2) shows that N_R is order dense in $\mathcal{U}$, this mean that R has zero carrier, finally note that $0 \leq R \leq \mathcal{F}$ holds.

From the previous theorem we know that $\ell_\sigma(\mathcal{U},\mathcal{V}) = \{0\}$ (means that $\ell_b(\mathcal{U},\mathcal{V}) = \ell_n(\mathcal{U},\mathcal{V})$) if and only if any nonzero operator between two Riesz spaces $\mathcal{U}$ and $\mathcal{V}$ has a nonzero carrier.
The following theorem explains some important relationships for order bounded operators.

Theorem 8 Alibrandi's and Burkinshaw (Aliprantis & Burkinshaw, 1983): For a pair of Riesz spaces $\mathcal{U}$ and $\mathcal{V}$ with $\mathcal{V}$ Dedekind complete, the following statements are equivalent:

1. Every order bounded operator from $\mathcal{U}$ into $\mathcal{V}$ is order continuous.
2. Every nonzero order bounded operator from $\mathcal{U}$ into $\mathcal{V}$ has a nonzero carrier.
3. The null ideal of every order bounded operator from $\mathcal{U}$ into $\mathcal{V}$ is a band.

The following result shows that when an operator is order continuous on a given ideal.

Theorem 9 Let $\mathcal{F}: \mathcal{U} \longrightarrow \mathcal{V}$ be a positive operator between two Riesz space with $\mathcal{V}$ Dedekind complete, and let U be an ideal of $\mathcal{U}$. Then the operator $\mathcal{F}$ is order (resp δ- order) continuous on U if and only if $\mathcal{F}_U$ is an order (resp δ-order) continuous operator.
Proof: first, the result of the "order continuous" case is proven. After that, the " σ-order continuous" case is proven in a similar way. the operator $\mathcal{F}_U$ is decided as follows:

$$\mathcal{F}_U(u) = \sup\{\mathcal{F}(v): v \in U \; and \; 0 \le v \le u\} ;$$

whereas $\mathcal{F}_U = \mathcal{F} \; \forall \, u \in U$, it is clear that if $\mathcal{F}_U$ is an order continuous operator, then $\mathcal{F}$ must be order continuous on U .
to prove the opposite direction, let $\mathcal{F}$ is order continuous on U, and $0 \le u_\alpha \uparrow u$ in $\mathcal{U}$, let $\mathcal{F}_U\,(u_\alpha) \uparrow s \le \mathcal{F}_U\,(u)$. Now fix $v \in U \bigcap [0,u]$. Then $0 \le v \wedge u_\alpha \uparrow v$ holds in U, and so $\mathcal{F}(v \wedge u_\alpha) \uparrow \mathcal{F}(v)$ holds in $\mathcal{V}$. From

$$\mathcal{F}(v \wedge u_\alpha) = \mathcal{F}_A(v \wedge u_\alpha) \le s \le \mathcal{F}_A(u) ,$$

It follows that $\mathcal{F}(v) \le s \le \mathcal{F}_U(u)$ holds $\forall v \in U \bigcap [0,u]$ Hence,

$$\mathcal{F}_U(u) = \sup \mathcal{F}(U \bigcap [0,u]) \le s \le \mathcal{F}_A\,(u),$$

and so $\mathcal{F}_U = s$ holds, proving that $\mathcal{F}_U$ is an order continuous operator.

CONCLUSION

To Sum up, we concluded the positive order continuous operator $\mathcal{F}: H \longrightarrow \mathcal{V}$ where H is order dense majorizing Riesz subspace of Riesz space $\mathcal{U}$ and $\mathcal{V}$ is Dedekind complete, it has a unique order continuous extension on all Riesz space $\mathcal{U}$, it defined as

$$\mathcal{F}(u) = \sup\{\mathcal{F}(\,h\,): h \in H\,, 0 \le h \le u\};\, u \in \mathcal{U}^+$$

The evidence of this result is as follows:
Since H majorizes $\mathcal{U}$ then:

located in $\mathcal{V}$ $\forall\, u \in \mathcal{U}^+$, now see:$E(u) = \sup\{\mathcal{F}\,(\,h\,): h \in H\,, 0 \le h \le u\}$
If $\{u_n\} \subseteq H$ when $0 \le u_\alpha \uparrow u$ then $\mathcal{F}(u_\alpha) \uparrow E(u_\alpha)$, and if $0 \le h \in H$ when $0 \le h \le u$, then $0 \le u_\alpha \wedge h \uparrow h$ satisfies in H, from the order continuity of $\mathcal{F} : H \to \mathcal{V}$ is:

$$\mathcal{F}(h) = sup\{\mathcal{F}(u_\alpha \wedge h)\} \le sup\{\mathcal{F}(u_\alpha)\} \le E(u)$$

This gives that $\mathcal{F}(u_\alpha) \uparrow E(u)$.

let $u, h \in \mathcal{U}^+$, choose nets $\{u_\alpha\}$ and $\{h_\beta\}$ of H^+ whereas $0 \le u_\alpha \uparrow u$ and $0 \le h_\beta \uparrow h$, this implies to that $0 \le u_\alpha + h_\beta \uparrow u + h$ holds in H^+, and so

$$.\mathcal{F}(u_\alpha) + \mathcal{F}(h_\beta) = \mathcal{F}(u_\alpha + h_\beta) \uparrow E(u + h)$$

from $\mathcal{F}(u_\alpha) \uparrow E(u)$ and $\mathcal{F}(h_\beta) \uparrow E(h) \Longrightarrow E(u + h) = E(u) + E(h)$ holds, and $E : \mathcal{U}^+ \to \mathcal{V}^+$ is additive operator, returning to theorem 1, $\mathcal{F}$ extends uniquely to $E: \mathcal{U} \longrightarrow \mathcal{V}$, this means E is an extension of $\mathcal{F}$.

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