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On Strongly Regular Relation of Canonical Hypergroup

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Abstract

Burris and Sankappanavar established a connection between congruence in group G (ring R) and a normal subgroup of G (ideal of ring R). In this paper in the same manner, the connection between strongly regular relation defined on canonical hypergroup and normal subcanonical hypergroup of canonical hypergroup is established.

Keywords: canonical hypergroup, subcanonical hypergroup, normal subcanonical hypergroup, regular relation, strongly regular relation, and quotient of canonical hypergroup **.**

INTRODUCTION

The French mathematician Marty proposed the idea of hyperstructure, and particularly the idea of hypergroup, in 1934 (Marty, 1934). There are basic definitions and theories concerning the hyperstructures previously (AbouElwan & Alderawe, 2023; Davvaz *et al*., 2023) Several fields of other disciplines can benefit from the use of hyperstructures. There have been numerous books and articles written about the use of hyperstructures in the study of geometry, hypergraphs, binary relations, lattices, fuzzy sets, etc (AbouElwan & Alderawe, 2023; Davvaz, 2012; Burris & Sankappanavar, 1981; Vougiouklis, 1994). Canonical hypergroup as a special kind of hypergroup is indeed a natural generalization of the concept of abelian group. This kind of hypergroup is a basic additive hyperstructure of many hyperstructures.

By applying a specific kind of equivalence relations, semihypergroup can be connected to semigroup, hypergroup to the group and canonical hypergroup to abelian group. These equivalence relations are called strongly regular relations. More exactly, by given (a semihypergroup, a hypergroup, and a canonical hypergroup) and by using a strongly regular relation on them, (a semigroup, a group, and the abelian group) respectively can be constructed from their quotient hyperstructures. (Corsini & Leoreanu, 2003)

MATERIALS AND METHODS

The definitions and examples from this section will be utilized throughout the paper.
A hyperoperation o on a non-empty set H is a mapping o: $H \times H \rightarrow P^*(H)$, $P^*(H)$ is the power set of H, $\varnothing \notin P^*(H)$. Moreover, the pair (H, o) is called a hyper-groupoid. For every A and B $\in$ P

*(H) and $x \in H$, the sets A o B, A o x and x o A are defined by A o B = $\cup$\{a o b | $a \in A$, $b \in B$\}, A o x = A o \{x\} and x o A = \{x\} o A.

A hypergroupoid (H, o) is called a semihypergroup if for all a, b, c of H, we have a o (b o c) = (a o b) o c, this means that $\bigcup_{u \in b \circ c} a \circ u = \bigcup_{v \in a \circ b} v \circ c$. A semihypergroup (H, o) is called a hypergroup if for every $a \in H$, we have a o H = H o a = H, that is called the reproduction axiom. A hypergroup (H, o) is called a commutative hypergroup if for all $a, b \in H$, we have a o b = b o a. A non-empty subset K of a hypergroup (H, o) is called a subhypergroup of H if K is a hypergroup under o. Several books have been written on hyperstructure theory (AbouElwan & Alderawe, 2022; Velrajan & Asokkumar, 2010; Vougiouklis, 1994).

Let (H, o) be a semihypergroup and R be an equivalence relation on H. If A, B are non-empty subsets of H, then $A\overline{\boldsymbol{R}}B$ means that

$\forall a \in A, \exists b \in B$ such that aRb, and $\forall b` \in B, \exists a` \in A$ such that a`Rb`.

Furthermore, $A\overline{\overline{\boldsymbol{R}}}B$ means that

$\forall a \in A, \forall b \in B$, we have aRb.

In addition, the equivalence relation R on H is said to be:

1) Regular on the left (on the right) if $\forall x \in H$, from aRb, it follows that $(x \circ a)\overline{\boldsymbol{R}}(x \circ b)$ $((a \circ x)\overline{\boldsymbol{R}}(b \circ x)$ respectively).
2) Strongly regular on the left (on the right) if $\forall x \in H$, from aRb, it follows that $(x \circ a)\overline{\overline{\boldsymbol{R}}}(x \circ b)$ $((a \circ x)\overline{\overline{\boldsymbol{R}}}(b \circ x)$respectively).
3) Strongly regular (Regular) if it is strongly regular (regular) on the right and on the left. (see 6).

Let (H, o) be a hypergroup, for an equivalence relation R on H, we use R(x) to denote the equivalence class of x to R and use H/R to denote the family of equivalence classes $\{R(x) \mid x \in H\}$ of R. The reader can find the proofs of the following two theorems in (BDavvaz & Leoreanu-Fotea, 2007; Davvaz *et al*., 2022).

Theorem 2.1. If (H, o) is a hypergroup (a semihypergroup) and R is a regular relation on H, then the quotient H/R is a hypergroup (a semihypergroup) under the operation defined by

$$R(x)\otimes R(y) =\{R(z) \mid z \in x \circ y\}.$$

Theorem 2.2. If (H, o) is a hypergroup (a semihypergroup) and R is a strongly regular relation on H, then the quotient H/R is a group (a semigroup) under the operation defined by

$$R(x)\otimes R(y) = R(z), \forall z \in x \circ y.$$

Definition 2.3. A canonical hypergroup (M, +) is a non-empty set M together with a hyperoperation + which satisfies the following axioms:

i. $\forall x, y \in M, x + y = y + x,$
ii. $\forall x, y, z \in M, x + (y + z) = (x + y) + z,$
iii. $\exists\, 0 \in M$ (called neutral element of M) such that

$$0 + x = \{x\} = x + 0, \forall x \in M,$$

iv. $\forall x \in M, \exists_1 -x \in M$ such that

$$0 \in x + (-x) \cap (-x) + x,$$

v. the reversibility axiom:
$\forall$ x, y, z $\in$ M, z $\in$ x + y $\Rightarrow$ y $\in$ – x + z and x $\in$ z + (– y).

Let (M, +) be a canonical hypergroup, a non-empty subset N of (M, +) is called a subcanonical hypergroup of M if (N, +) is a canonical hypergroup itself. Equivalently, $x - y \subseteq N$, $\forall$ x, y $\in$ N.

In particular, $\forall$ x $\in$ N, $x - x \subseteq N$. Since $0 \in x - x$, it follows that 0 $\in$ N. Moreover, N is said to be normal if $x + N - x \subseteq N$, for all x $\in$ M. In addition, a subcanonical hypergroup N of M is called a subgroup of M if (N, +) is a group, that is, if x + y is a singleton set for all x, y $\in$ N.

Example 2.4. Consider the set M ={0, a, b}. Define a hyperaddition + on M as in the following table

+	0	*A*	*b*	*c*
0	{0}	{*a*}	{*b*}	{*c*}
a	{*a*}	{0, *b*}	{*a, c*}	{*b*}
b	{*b*}	{*a, c*}	{0, *b*}	{*a*}
c	{*c*}	{*b*}	{*a*}	{0}

Then, (M, +) is a canonical hypergroup, {0, b} is a subcanonical hypergroup of M, and {0, c} is a subgroup of M.

Remark 2.5. If N be a subcanonical hypergroup of a canonical hypergroup (M, +), then the quotient is M/N = {x + N | x $\in$ M},

where x + N = {x + n | n $\in$ N}, we will use $\overline{\boldsymbol{x}}$ instead of x + N.

Theorem 2.6.[11] If N be a subcanonical hypergroup of a canonical hypergroup (M, +). Then M/N is a canonical hypergroup with respect to the following hyperoperation

$$(x + N)\oplus(y + N) =\{z + N \mid z \in x + y\}, \text{ for all } x + N,\ y + N \in M/N.$$

Proof. Let $x_1, y_1, x_2, y_2 \in M$ such that $\overline{\boldsymbol{x}}_{\boldsymbol{1}} = \overline{\boldsymbol{x}}_{\boldsymbol{2}}$ and $\overline{\boldsymbol{y}}_{\boldsymbol{1}} = \overline{\boldsymbol{y}}_{\boldsymbol{2}}$ then $x_2 \in x_1 + N$ and $y_2 \in y_1 + N$. Let $z_2 \in x_2 + y_2 \subseteq (x_1 + N) + (y_1 + N)$. Since M is commutative, $z_2 \in z_1 + n$ for some $z_1 \in x_1 + y_1$ and for some n $\in$ N. That is, $z_2 + N = z1 + N$. Hence,

$$\overline{\boldsymbol{x}}_{\boldsymbol{2}}\oplus\overline{\boldsymbol{y}}_{\boldsymbol{2}} \subseteq \overline{\boldsymbol{x}}_{\boldsymbol{1}}\oplus\overline{\boldsymbol{y}}_{\boldsymbol{1}}.$$

Also, since $x_1 \in x_2 + N$ and $y_1 \in y_2 + N$, by a similar argument, we get,

$$\overline{\boldsymbol{x}}_{\boldsymbol{1}}\oplus\overline{\boldsymbol{y}}_{\boldsymbol{1}} \subseteq \overline{\boldsymbol{x}}_{\boldsymbol{2}}\oplus\overline{\boldsymbol{y}}_{\boldsymbol{2}}.$$

Hence, $\overline{\boldsymbol{x}}_{\boldsymbol{1}}\oplus\overline{\boldsymbol{y}}_{\boldsymbol{1}} = \overline{\boldsymbol{x}}_{\boldsymbol{2}}\oplus\overline{\boldsymbol{y}}_{\boldsymbol{2}}$. Thus, $\oplus$ is well defined.

Let $\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{z}} \in M/N$. If $\overline{\boldsymbol{u}} \in (\overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{y}})\oplus\overline{\boldsymbol{z}}$, then $\overline{\boldsymbol{u}} \in \overline{\boldsymbol{p}}\oplus\overline{\boldsymbol{z}}$ for some $\overline{\boldsymbol{p}} \in \overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{y}}$. That is, $\overline{\boldsymbol{u}} = \overline{\boldsymbol{a}}$ for some a $\in$ p + z. Also $\overline{\boldsymbol{p}} = \overline{\boldsymbol{b}}$ for some b $\in$ x + y.

Now, $a \in p + z \subseteq b + N + z = b + z + N$. That is, $a \in v + N$ for some $v \in b + z \subseteq (x + y) + z = x + (y + z)$. So, $v \in x + t$ for some $t \in y + z$. This means that, $\overline{\boldsymbol{a}} = \overline{\boldsymbol{v}}$ and $\overline{\boldsymbol{v}} \in \overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{t}}$. Since $\overline{\boldsymbol{t}} \in \overline{\boldsymbol{y}}\oplus\overline{\boldsymbol{z}}$, we have

$$\overline{\boldsymbol{u}} = \overline{\boldsymbol{a}} = \overline{\boldsymbol{v}} \in \overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{t}} \subseteq \overline{\boldsymbol{x}}\oplus(\overline{\boldsymbol{y}}\oplus\overline{\boldsymbol{z}}).$$

This means that, $\overline{\boldsymbol{u}} \in \overline{\boldsymbol{x}}\oplus(\overline{\boldsymbol{y}}\oplus\overline{\boldsymbol{z}})$. Hence,

$$(\overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{y}})\oplus\overline{\boldsymbol{z}} \subseteq \overline{\boldsymbol{x}}\oplus(\overline{\boldsymbol{y}}\oplus\overline{\boldsymbol{z}}).$$

Similarly, we get

$$\overline{\boldsymbol{x}}\oplus(\overline{\boldsymbol{y}}\oplus\overline{\boldsymbol{z}}) \subseteq (\overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{y}})\oplus\overline{\boldsymbol{z}}. \text{ Hence, } (\overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{y}})\oplus\overline{\boldsymbol{z}} = \overline{\boldsymbol{x}}\oplus(\overline{\boldsymbol{y}}\oplus\overline{\boldsymbol{z}}).$$

Thus, the hyperoperation $\oplus$ is associative.

Consider the element $\overline{\boldsymbol{0}} = 0 + N \in M/N$. Now, for any $x \in M$, we have

$$\overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{0}} = \{\overline{\boldsymbol{z}} \mid z \in x + 0\} = \overline{\boldsymbol{x}}.$$

Similarly, $\overline{\boldsymbol{0}}\oplus\overline{\boldsymbol{x}} = \overline{\boldsymbol{x}}$. Thus, $\overline{\boldsymbol{0}}$ is the zero element of M/N.

Let $x \in M$, then $\overline{\boldsymbol{x}}\oplus(-\overline{\boldsymbol{x}}) = \{\overline{\boldsymbol{z}} \mid z \in x + (-x) = x - x\}$. Since $\overline{\boldsymbol{0}} \in x - x$, we get, $\overline{\boldsymbol{0}} \in \overline{\boldsymbol{x}}\oplus(-\overline{\boldsymbol{x}})$. Similarly, $\overline{\boldsymbol{0}} \in (-\overline{\boldsymbol{x}})\oplus\overline{\boldsymbol{x}}$. Let $\overline{\boldsymbol{x}} \in M/N$, and suppose that $\overline{\boldsymbol{y}} \in M/N$ such that $\overline{\boldsymbol{0}} \in \overline{\boldsymbol{y}}\oplus\overline{\boldsymbol{x}}$, then $\overline{\boldsymbol{0}} = \overline{\boldsymbol{a}}$, where $a \in y + x$. That is, $y \in a - x \subseteq N - x$, and hence $\overline{\boldsymbol{y}} = -\overline{\boldsymbol{x}}$. Thus, the element $\overline{\boldsymbol{x}} \in M/N$ has a unique inverse $-\overline{\boldsymbol{x}} \in M/N$. Suppose that $\overline{\boldsymbol{z}} \in \overline{\boldsymbol{x}}\oplus\overline{\boldsymbol{y}}$, then $\overline{\boldsymbol{z}} = \overline{\boldsymbol{a}}$, where $a \in x + y$. This implies,

$x \in a - y \subseteq z + N - y$. That is, $x \in r + N$, where $r \in z - y$. Thus, $\overline{\boldsymbol{x}} = \overline{\boldsymbol{r}} \in \overline{\boldsymbol{z}}\oplus(-\overline{\boldsymbol{y}})$. Similarly, we can show that $\overline{\boldsymbol{y}} \in (-\overline{\boldsymbol{x}})\oplus\overline{\boldsymbol{z}}$. Since M is commutative, it is obvious that M/N is also commutative. Thus, M/N is a canonical hypergroup. ■

Theorem 2.7.[11] Let (M, +) be a canonical hypergroup, and let N be a normal subcanonical hypergroup of M. Then, (M/N, $\oplus$) is an abelian group.

RESULTS

Definition 3.1. Let $(M, +)$ be a canonical hypergroup, and ρ be an equivalence relation on M, then ρ is called:

1) Regular if for all $a, b \in M$, $a\rho b$ implies that for every $x \in M$, for every $u \in a + x$ there exists $v \in b + x$ such that $u\rho v$ and for every $v' \in b + x$ there exists $u' \in a + x$ such that $u'\rho v'$.
2) Strongly regular if for all $a, b \in M$, $a\rho b$ implies that for every $x \in M$, for every $u \in a + x$ and for every $v \in b + x$ one has $u\rho v$.

Proposition 3.2.[6] Let $(M, +)$ is a canonical hypergroup, and let N be a normal subcanonical hypergroup of M. Then, for all $x, y \in N$, the following are equivalent:

i. $y \in x + N$,

ii. $x - y \subseteq N$,

iii. $(x - y) \cap N \neq \emptyset$.

Proof.

($i \Longrightarrow ii$). Since $y \in x + N$, we have $y - x \subseteq x + N - x$, and since N is normal subcanonical hypergroup of M, we get $x + N - x \subseteq N$. Thus, $y - x \subseteq N$. That is, $-(y - x) \subseteq N$, and hence $x - y \subseteq N$.

($ii \Longrightarrow iii$). Is obvious.

($iii \Longrightarrow i$). Since $(x - y) \cap N \neq \emptyset$, there exists $a \in x - y$ and $a \in N$. Therefore, $-y + x \subseteq -y + a + y \subseteq N$. If $z \in -y + x$, then $z \in N$. Therefore, $-y \in z - x$. That is, $y \in x - z \subseteq x + N$. ■

Now, if $(M, +)$ is a canonical hypergroup, then we can establish the following connection between regular relations on M and subcanonical hypergroups of M, and establishe a similar connection between strongly regular relations on M and normal sub-canonical hypergroups of M as follows:

i. If ρ is a regular relation on M. Then the equivalence class $\rho(0)$ is a subcanonical hypergroup of M, where 0 is a neutral element of M. For $a, b \in M$, we have

$a\rho b$ iff $a \in b + \rho(0)$.

ii. If ρ is a strongly regular relation on M. Then the equivalence class $\rho(0)$ is a normal subcanonical hypergroup of M, where 0 is a neutral element of M. For $a, b \in M$, we have

$a\rho b$ iff $a - b \subseteq \rho(0)$.

Theorem 3.3. Let $(M, +)$ be a canonical hypergroup, and let N be a normal subcanonical hypergroup of M. If a, b are elements in M, then the binary relation ρ defined on M by:

$$a\rho b \quad \text{iff} \quad a - b \subseteq N,$$

is a strongly regular on M with $\rho(0) = N$.

Proof. Let $a, b, c \in M$. Clearly, $a \in a + 0$, implies

$$a - a \subseteq a + 0 - a \subseteq a + N - a \subseteq N.$$

So ρ is reflexive. Also, $a - b \subseteq N$ if and only if $b - a \subseteq N$. So ρ is symmetric. For transitivity, if $a - b \subseteq N$ and $b - c \subseteq N$ then by normality of N, we have

$$a - b + b - c = a - b + 0 + b - c \subseteq a + N - c \subseteq a - c + N \subseteq N,$$

$$a - c \subseteq N.$$

Thus ρ is an equivalence relation on M.

Next, to prove that the equivalence relation ρ is a strongly regular on M, suppose that $a\rho b$ then $a - b \subseteq N$, let $x \in M$, if $u \in a + x$ and $v \in b + x$, then

$u - v \subseteq a + x - (b + x) = a + x - (x + b) = a + (x - x) - b \subseteq a + (x + 0 - x) - b \subseteq a + N - b \subseteq N$, so $u\rho v$, thus

$$(a + x)\bar{\bar{\rho}}(b + x).$$

Hence ρ is a strongly regular relation on M.

Now, to prove that $\rho(0)$ is a normal subcanonical hypergroup of M. Let $a, b \in \rho(0)$, then $a\rho 0$ and $b\rho 0$, since ρ is a strongly regular relation on M. this imply that

$$(a + b)\bar{\bar{\rho}}(0 + 0),$$

this means that, for all $u \in a + b$ and $0 \in 0 + 0$, we have $u\rho 0$, it follows that

$$u \in \rho(0), \text{ so } a + b \subseteq \rho(0).$$

Since $a \in \rho(0)$ it follows that $-a \in \rho(0)$. Therefore $\rho(0)$ is a subcanonical hypergroup of M. For normality of $\rho(0)$, let $a \in \rho(0)$ and $x \in M$, then

$$(x + a)\bar{\bar{\rho}}(x + 0),$$

this implies that $\quad (x + a)\bar{\bar{\rho}}x,$

it follows that $\quad (x + a - x)\bar{\bar{\rho}}(x - x),$

then $(x + a - x)\bar{\bar{\rho}}0$, since $0 \in x - x$.

Therefore, $x + a - x \subseteq \rho(0)$.

Thus $\rho(0)$ is a normal subcanonical hypergroup of M.

Finally, to prove that $\rho(0) = N$, if $a \in \rho(0)$ then $a\rho 0$ implies $a - 0 \subseteq N$, so $a \in N$, thus $\rho(0) \subseteq N$. Conversly, if $a \in N$ then $a - 0 \subseteq N$, it follows that $a\rho 0$ implies $a \in \rho(0)$, thus $N \subseteq \rho(0)$. Hence, $\rho(0) = N$. ■

CONCLUSION

By strongly regular relation defined on a canonical hypergroup M, the equivalence class $\rho(0)$ is exactly a normal subcanonical hypergroup of M.

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