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On Menger Spaces in Generalized Topology

Assakta K. Bashier

***Corresponding author:** alsaktamohammed@bwu.edu. Department of Mathematics, Faculty of Science, Bani Waleed University, Libya.

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Abstract

We introduce new types of covering properties in generalized topology, namely; λ-Menger and λ-uniformly Menger spaces, and investigate their fundamental properties. To achieve this, we replace open sets in the definition of the standard Manger spaces with λ-open sets of generalized topological spaces. The results show that the λ-Menger property is stronger than the Menger property. Additionally, λ-Menger spaces are preserved when forming subspaces and countable unions. We also characterize λ-uniformly Menger spaces and study their relationship with λ-Menger spaces. Examples are given to further illustrate our results.

Keywords: Generalized topological space; λ-Menger space; λ-uniform space

INTRODUCTION

In this paper, we investigate the Menger covering property of generalized topological spaces. We introduce the concepts of λ-Menger spaces and λ-uniformly Menger spaces and investigate some of their characteristics. Generalized topological spaces, in the sense of A. Csàszàr, was introduced in (Csàszàr, 2002). A collection λ of subsets of a set X is called a generalized topology (GT) if it satisfies the following:
(1) $\emptyset \in \lambda$,
(2) λ is closed under arbitrary unions.
The elements of λ are generalized open sets (λ-open sets) and their complements are λ-closed sets. In generalized topology, the condition that the whole space is a λ-open set is dropped. A generalized topological space (GTS) is a pair (X, λ), where X is a non-empty set and λ is a generalized topology on X. If $X \in \lambda$, then (X, λ) is called a strong GTS (λ-space). Every topological space is a λ-space but the converse is not true (Csàszàr, 2002). In topology, several types of generalized open sets were introduced. For example, semi-open sets (Levine, 1963), α-open sets (Njåstad, 1965), Pre-open sets (Mashhour et al., 1982), and β-open sets (Abd El-Monsef et al., 1983). However, the concept of λ-open sets contains all these classes of open sets (Csàszàr, 2002). Various topological notions were examined in the context of generalized topology, for instance, generalized homotopy (Bashier, 2022), generalized separation axioms (Makai et al., 2016), and λ-compactness (Sarsak, 2013). In this work, we investigate the Menger property and its uniform version in the setting of generalized topology by using covers whose members are λ-open sets.

The Menger property is one of the classical selection principles which was first introduced by K. Menger (Menger, 1924). A topological space (X,τ) is called Menger (or has Menger property) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each n and $\bigcup_{n\in N} \mathcal{V}_n$ is an open cover of X. Further details on selection principles can be found in (Kočinac, 2020) and (Scheepers, 1996) and references therein.

There are generalizations of the concept of Menger spaces in the literature, for example, uniformly Menger (Kočinac, 2003), Semi-Menger (Sabah et al., 2016), α-Menger (Kočinac, 2019), Pre-Menger (Tyagi et al., 2021), and β-Menger (Kule, 2022). Our new generalized structures, the λ-Menger and the λ-uniformly Menger spaces, extend and complement previous work.

PRELIMINARIES

Through this article, a λ-space (X,λ) (or just X) means a strong GTS. The entourages of the diagonal are used in our approach to uniform spaces, the reader is referred to [(Engelking, 1989), Chapter 8] for unexplained notation or terminology.

Definition 2.1 (Csàszàr, 2007) Let (X,λ) be a GTS. A base for a GT λ, denoted by $\mathcal{B}$, is a collection of subsets of X with $\emptyset \in \mathcal{B}$ such that $\lambda = \{\bigcup_{i\in I} B_i : B_i \in \mathcal{B}\}$.

Definition 2.2 (Csàszàr, 2002) Let (X,λ_1) and (Y,λ_2) be two GTS's. A function $f : (X,\lambda_1) \to (Y,\lambda_2)$ is called (λ_1,λ_2)-continuous if $m \in \lambda_2$ implies $f^{-1}(m) \in \lambda_1$.

Definition 2.3 (Sarsak, 2013) Let A be a nonempty subset of (X,λ). The generalized subspace is a pair (A,λ_A), where $\lambda_A = \{U \cap A : U \in \lambda\}$ is the GT on A.

Observe that, if (X,λ) is a λ-space, then (A,λ_A) is a λ_A-space [(Sarsak, 2013), Remark

2.12].

Definition 2.4 (Sarsak, 2013) A λ-space (X,λ) is called λ-compact (λ-Lindelöf, respectively) if any cover of X by λ-open sets has a finite (countable, respectively) subcover.

Definition 2.5 (Dey et al., 2022) Let X be a non-empty set. A non-empty family $\mathbb{U}_\lambda$ of subsets of $X \times X$ is called a λ-uniformity on X if:

(1) $U \in \mathbb{U}_\lambda$ then $\Delta \subseteq U$, where $\Delta = \{(x,x) : x \in X\}$ is the diagonal on $X \times X$.
(2) $U \in \mathbb{U}_\lambda$ and $V \supseteq U$ then $V \in \mathbb{U}_\lambda$.
(3) $U \in \mathbb{U}_\lambda$ then there exists some $V \in \mathbb{U}_\lambda$ such that $V \circ V \subseteq U$.

The pair $(X,\mathbb{U}_\lambda)$ is called a λ-uniform space.
Recall that, given $U \in \mathbb{U}_\lambda$, $x \in X$ and $A \subset X$, then $U[x] = \{y \in X : (x,y) \in U\}$ and $U[A] = \bigcup_{x\in A} U[x]$ (Engelking, 1989).

Theorem 2.1 [(Dey et al., 2022), Theorem 2.10] Let $(X,\mathbb{U}_\lambda)$ be a λ-uniform space, and let $\tau(\mathbb{U}_\lambda)$ be a collection of subsets of X defined as: A subset $G \in \tau(\mathbb{U}_\lambda)$ if and only if for every $x \in G$, there exists some $U_x \in \mathbb{U}_\lambda$ such that $U_x[x] \subseteq G$.Then $\tau(\mathbb{U}_\lambda)$ is a strong GT on X.

The GT $\lambda = \tau(\mathbb{U}_\lambda)$ is called the generalized topology on X induced by the λ-uniformity $\mathbb{U}_\lambda$.

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Before we start the main results, we recall the definition of generalized covers. A λ-open cover of (X,λ) is a collection $\mathcal{U}$ of subsets of X such that the elements of $\mathcal{U}$ are λ-open sets and $X \subseteq \bigcup \mathcal{U}$ (Thomas & John, 2012). A λ-open subcover of $\mathcal{U}$ is a sub-collection $\mathcal{V} \subset \mathcal{U}$ which itself

is a λ-open cover (Thomas & John, 2012).
Definition 3.1. A λ-space (X,λ) is called a λ-Menger space (or has the λ-Menger property) if for any sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of λ-open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$.

Example 3.1. Let $\mathbb{R}$ be the set of real numbers. Consider the standard strong GTS $(\mathbb{R},\lambda_s)$ introduced in (Csàszàr, 2007), where λ_s has a base set given by
$\mathcal{B} = \{(-\infty,s) : s \in \mathbb{R}\} \cup \{(t,\infty) : t \in \mathbb{R}\}$
We claim that $(\mathbb{R},\lambda)$ is a λ_s-Menger space. To prove this, observe that the λ_s-open sets take one of the following forms: $\emptyset$, $\mathbb{R}$, $(-\infty,s)$, (t,∞), $(-\infty,s) \cup (t,\infty)$ where $s \le t$ and $s,t \in \mathbb{R}$. Using the fact that any open interval can be written as the union of an increasing sequence of compact sets, let

$$(-\infty,s) = \bigcup_{n\in\mathbb{N}} [s-n, s-1/n],$$

$$(t,\infty) = \bigcup_{n\in\mathbb{N}} [t+1/n, t+n].$$

Therefore, for any sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of λ_s-open covers of $\mathbb{R}$ and for each $n \in \mathbb{N}$, there is a finite sub-collection $\mathcal{V}_n \subseteq \mathcal{U}_n$, which covers the compact intervals $[s-n, s-1/n] \cup [t+1/n, t+n]$.
It easily follows that $\bigcup_{n\in N} \mathcal{V}_n$ is a λ_s-open cover of $\mathbb{R}$.

Example 3.2. Let $\mathbb{R}$ be the set of real numbers and $\lambda = \{G \subseteq \mathbb{R} : 0 \in G\} \cup \{\emptyset\}$. Then $(\mathbb{R},\lambda)$ is a λ-space. The set $\{\{0,x\} : x \in \mathbb{R}\}$ is a λ-open cover of $\mathbb{R}$ which does not contain a countable subcover. Therefore, $(\mathbb{R},\lambda)$ is not λ-Lindelöf, and hence, it is not λ-Menger.

Remark 3.1. Evidently, every λ-compact space is λ-Menger, and every λ-Menger space is λ-Lindelöf. The converse is not true for both cases.

Example 3.3. There is a λ-Menger space that is not λ-compact. Consider the λ-Menger space in Example 3.1. The set $\{(-\infty,n) : n \in \mathbb{N}\}$ is a λ_s-open cover of $\mathbb{R}$ that does not have a finite λ_s-subcover. Hence $(\mathbb{R},\lambda_s)$ is not λ_s-compact.

Example 3.4. There is a λ-Lindelöf space that is not λ-Menger. Let (X,λ) be a λ-space where $X = [0,1) \subset \mathbb{R}$, and λ has as a base given by
$\mathcal{B} = \{[0,r) : r \in [0,1)\} \cup \{[r,1) : r \in [0,1)\}$.
The λ-open sets take one of the following forms: $\emptyset$, X, $[0,r)$, $[r,1)$, $[0,r) \cup [s,1)$ where $r,s \in [0,1)$ and $r \le s$. Using similar arguments as in [(Engelking, 1989), Example 3.8.14], we can easily see that every λ-open cover of X by basis elements has a countable subcover; hence, X is a λ-Lindelöf space. On the other hand, X is not λ-Menger. To prove this, let $a,b \in X$ such that $a < b$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ-open covers of X consisting of intervals $[a,b)$, where each cover $\mathcal{U}_{n+1}$ is obtained by dividing each interval in $\mathcal{U}_n$ into smaller ones. For each $n \in \mathbb{N}$, choose a finite $\mathcal{V}_n \subset \mathcal{U}_n$. Since $\mathcal{V}_n$ is finite, pick one element $[a_n,b_n) \in \mathcal{U}_n$ such that $[a_n,b_n) \notin \mathcal{V}_n$. Then, there is one point x in the intersection of the intervals $[a_n,b_n)$ such that $x \notin \mathcal{V}_n$ for each n. Therefore, $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is not a λ-open cover of X.

Remark 3.2. If λ is not a strong GT on X, then (X,λ) is λ-compact and therefore it is λ-Menger.

Lemma 3.1. If (X, λ) is a finite GTS, then X is λ-Menger.

Proof. Every finite GTS (X, λ) is λ-compact [(Thomas & John, 2012), Theorem 3.5], hence, it is λ-Menger. □

Let the collection of all semi-open (Levine, 1963) (Pre-open (Mashhour et al., 1982), α-open (Njåstad, 1965), β-open (Abd El-Monsef et al., 1983), respectively) subsets of a topological space (X, τ) be denoted by $SO(X)$ ($PO(X)$, $\alpha O(X)$, $\beta O(X)$, respectively). If $\lambda = SO(X)$ ($PO(X)$, $\alpha O(X)$, $\beta O(X)$, respectively), then (X, λ) is a λ-space (Csàszàr, 2002). Evidently, we have:

Proposition 3.1. If (X, τ) is a topological space and $\lambda = SO(X)$ (resp. $PO(X)$, $\alpha(X)$, $\beta(X)$), then the following are equivalent:
(1) (X, λ) is λ-Menger.
(2) (X, τ) is semi-Menger (Sabah et al., 2016) (pre-Menger (Tyagi et al., 2021), α-Menger (Kočinac, 2019), β-Menger (Kule, 2022), respectively).

Theorem 3.1. Let λ_1 and λ_2 be two strong GT's on a set X with $\lambda_1 \subset \lambda_2$. If (X, λ_2) is λ_2-Menger, then (X, λ_1) is λ_1-Menger.

Proof. Let (X, λ_1) and (X, λ_2) be two GTS's such that $\lambda_1 \subset \lambda_2$. Suppose that (X, λ_2) is λ_2-Menger. Since $\lambda_1 \subset \lambda_2$, then any sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of λ_2-open covers of (X, λ_2) will also cover (X, λ_1). Applying the λ_2-Menger property of (X, λ_2), we can find a sequence $\mathcal{V}_n \subset \mathcal{U}_n$ of finite subsets whose union covers (X, λ_1). Hence, (X, λ_1) is λ_1-Menger. □

Proposition 3.2. Let (X, τ) be a topological space. Every λ-Menger space is Menger.

Proof. For any topological space (X, τ), let λ stands for any of the families $\alpha O(X)$, $PO(X)$, $SO(X)$, or $\beta(X)$. We have that
$\tau \subset \alpha O(X) \subset PO(X) \subset \beta O(X)$,
and
$\tau \subset \alpha O(X) \subset SO(X) \subset \beta O(X)$.
It follows by Theorem 3.1 that if (X, λ) is a λ-Menger space, then (X, τ) is a Menger space. □

Theorem 3.2. Let X_n be a subset of a λ-space (X, λ), where $n \in \mathbb{N}$. If $X = \bigcup_{n\in\mathbb{N}} X_n$ and X_n is a λ_{X_n}-Menger space for each n, then (X, λ) is a λ-Menger space.

Proof. Let $X_n \subseteq X$ with $X = \bigcup_{n\in\mathbb{N}} X_n$. Assume that the generalized subspace (X_n, λ_{X_n}) is λ_{X_n}-Menger for each $n \in \mathbb{N}$. Now, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ-open covers of X. Since $X = \bigcup_{n\in\mathbb{N}} X_n$, then $\mathcal{U}_n$ will also be a sequence of λ-open covers of X_n for each n. But X_n is λ_{X_n}-Menger, so for each $n \in \mathbb{N}$, there exists a finite sub-collection $(\mathcal{V}_m^n : m \in \mathbb{N})$ of $(\mathcal{U}_n : n \in \mathbb{N})$ covering each X_n with $\bigcup_{m\in\mathbb{N}} \mathcal{V}_m^n$ covers X_n. Set $\mathcal{V}_n = \bigcup_{m\in\mathbb{N}} \{\mathcal{V}_m^n : m, n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a λ-open cover of (X, λ). □

Theorem 3.3. Let (A, λ_A) be a generalized subspace of a λ-space (X, λ). If X is a λ-Menger space, then A is a λ-Menger space provided that A is λ-closed and λ-open in X.

Proof. Let (A, λ_A) be a λ-open and λ-closed generalized subspace of a λ-Menger space (X, λ). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ-open covers of (A, λ_A). By Definition 2.3, every λ_A-open set of the

λ-open and λ-closed A of X is the intersection of a λ-open set of X with A. Therefore, for each $n \in \mathbb{N}$ and for each $U \in \mathcal{U}_n$, there exists a λ-open set $M(U,n)$ in X such that $U = A \cap M(U,n)$. Let $\mathcal{M}_n = \{M(U,n): U \in \mathcal{U}_n\} \cup \{X \backslash A\}$, $n \in \mathbb{N}$. Thus, $(\mathcal{M}_n: n \in N)$ is a sequence of λ-open covers of X. Applying the λ-Menger property of X, there exists a finite subset $\mathcal{W}_n$ of $\mathcal{M}_n$ for each $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$covers X by λ-open sets. Let $\mathcal{V}_n = \{U: M(U,n) \in W_n\}$ for each n. The sequence $(\mathcal{V}_n: n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n: n \in \mathbb{N})$ that (A, λ_A) is a λ-Menger space. □

Theorem 3.3 generalizes several results in the literature. For example, in a topological space (X,τ), if we take $\lambda = PO(X)$ then we get Theorems 3.5 of (Tyagi et al., 2021). If $\lambda = \beta O(X)$, we get Proposition 4.3 of (Kule, 2022). If $\lambda = \alpha O(X)$, we get Proposition 2.4 of (Kočinac, 2019).

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Definition 4.1. A λ-uniform space $(X, \mathbb{U}_\lambda)$ is said to be λ-totally bounded if for each $U \in \mathbb{U}_\lambda$, there exists a finite subset A of X such that $U[A] = X$.

Definition 4.2. Let $(X, \mathbb{U}_\lambda)$ be a λ-uniform space. We say that X is λ-uniformly Menger space (or has the λ-uniform Menger property) if for any sequence $(U_n: n \in \mathbb{N})$ of elements of $\mathbb{U}_\lambda$, there exists a sequence $(A_n: n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n\in N} U_n[A_n]$.

Theorem 4.1. Let $(X, \mathbb{U}_\lambda)$ be a λ-uniform space. Then X is λ-uniformly Menger if and only if for each sequence $(U_n: n \in \mathbb{N})$ in $\mathbb{U}_\lambda$, there exists a sequence $(V_n: n \in \mathbb{N})$ of finite subsets of X such that for each $n \in \mathbb{N}$ and for each $V \in V_n$, we have $V \times V \subseteq U_n$ and $\bigcup_{n\in N} V_n$ is a λ-open cover of X.

Proof: Let $(U_n: n \in \mathbb{N})$be a sequence in $\mathbb{U}_\lambda$. Suppose that X is λ-uniformly Menger. By definition, there is a sequence $(A_n: n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n\in\mathbb{N}} U_n[A_n]$. For each $n \in \mathbb{N}$ and for each $A \in A_n$, set $V_1 = U_n[A_1]$, $V_2 = U_n[A_2]$, … , $V_n = U_n[A_n]$. Then $(V_n: n \in \mathbb{N})$ is a sequence of finite subsets of X such that $X \subseteq \bigcup_{n\in N} V_n$. Also, $V \subset U_n[A]$ for each $V \in V_n$and for some $A \in A_n$, therefore $V \times V \subseteq U_n$.
On the other hand, let $(U_n: n \in \mathbb{N})$ be a sequence in $\mathbb{U}_\lambda$. Let $(V_n: n \in \mathbb{N})$ be a sequence of finite subsets of X that satisfies the conditions of the second part of the theorem. For each $n \in \mathbb{N}$ and for each $\in V_n$, choose a point x_V^n in V_n such that $x \in U_n[x_V^n]$ for each $x \in X$. Define a sequence $(A_n: n \in \mathbb{N})$ as $A_n = \{x_V^n : x_V^n \in V_n\}$. Then each A_n is a finite subset of X such that $X = \bigcup_{n\in N} U_n[A_n]$. Hence, X is λ-uniformly Menger. □

Proposition 4.1. Let $\mathbb{U}_\lambda$and $\mathbb{V}_\lambda$ be two λ-uniformities on a set X such that $\mathbb{U}_\lambda \subseteq \mathbb{V}_\lambda$. If $(X, \mathbb{V}_\lambda)$ is λ-uniformly Menger, then $(X, \mathbb{U}_\lambda)$ is also λ-uniformly Menger.

Theorem 4.2. Let (X, λ) be a λ-space, where $\lambda = \tau(\mathbb{U}_\lambda)$ the GT is induced by the λ-uniformity $\mathbb{U}_\lambda$. If (X, λ) is a λ-compact space, then $(X, \mathbb{U}_\lambda)$ is λ-totally bounded space.

Theorem 4.3. Let (X, λ) be a λ-space, where $\lambda = \tau(\mathbb{U}_\lambda)$ is the GT induced by the λ-uniformity $\mathbb{U}_\lambda$. If (X, λ) is a λ-Menger space, then $(X, \mathbb{U}_\lambda)$ is a λ-uniformly Menger space.
Proof. Let $(U_n: n \in \mathbb{N})$ be a sequence in $\mathbb{U}_\lambda$. The λ-uniformity $\mathbb{U}_\lambda$ generates a GT $\lambda = \tau(\mathbb{U}_\lambda)$, where $G \in \lambda$ if and only if for every $x \in X$, there is some $U_x \in \mathbb{U}_\lambda$ such that $U_x[x] \subseteq G$ [5, Theorem 2.10]. Suppose that (X, λ) is a λ-Menger space, where $\lambda = \tau(\mathbb{U}_\lambda)$ is the GT induced by the λ-uniformity $\mathbb{U}_\lambda$. Then, by Definition 3.1, for each sequence $(\mathcal{U}_n: \mathrm{n} \in \mathbb{N})$ of λ-open covers of X, there exists a sequence of finite subsets $\mathcal{V}_n$ of $\mathcal{U}_n$such that $\mathrm{X} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$. Let $V \in \mathcal{V}_n$. Since V is a λ-open set,

there is some $U_x \subset U_n \in \mathbb{U}_\lambda$ such that $U_x[x] \subseteq V$. Choose a point $x_V \in V$ and let $A_n = \{x_V : V \in \mathcal{V}_n\}$. Then for each n and for each $V \in \mathcal{V}_n$ we have $V \subset U_n[x_V]$ and therefore, by Theorem 4.1, the sequence $(A_n : n \in \mathbb{N})$ witnesses for $(U_n : n \in \mathbb{N})$ that X is a λ- uniformly Menger space. □

Remark 4.1. The converse of Theorem 4.3 is not true. Consider the following example.

Example 4.1. cf. [(Kočinac, 2003), Note 3] There is a λ-uniform space $(X, \mathbb{C}_\lambda)$ which is λ-uniformly Menger, but the λ-space $(X, \tau(\mathbb{C}_\lambda))$ is not λ-Menger. To prove this, recall that every λ-Menger space is λ-Lindelöf (Remark 3.1). Let (X, λ) be a λ-Tychonoff space (Makai et al., 2016), which is not λ-Lindelöf. Hence, X is not λ-Menger.

On the other hand, consider $(\mathbb{R}, \lambda_s)$ as defined in Example 3.1. It is shown that $(\mathbb{R}, \lambda_s)$ is a λ_s-Tychonoff space (Makai et al., 2016). Let $C_{\lambda,\lambda_s}(X) = \{f : (X, \lambda) \to (\mathbb{R}, \lambda_s)\}$ be the set of all (λ, λ_s)-continuous and bounded functions. The authors (Gupta & Sarma, 2015) introduced a topology τ on $C_{\lambda,\lambda_s}(X)$, which has a sub-base given by the set $S_{\lambda,\lambda_s} = \{(U, V) : U \in \lambda, V \in \lambda_s\}$,
where
$(U, V) = \{f \in C_{\lambda,\lambda_s}(X) : f(U) \subseteq V\}$.

Now, using $(C_{\lambda,\lambda_s}(X), \tau)$ in arguments similar to Example 8.1.19 and Example 8.3.4 in (Engelking, 1989), we can construct a λ-uniformity $\mathbb{C}_\lambda$ on X, which generates the original GT on X. Moreover, $(X, \mathbb{C}_\lambda)$ is λ-totally bounded and thus, λ-uniformly Menger.

CONCLUSION

The present paper deals with the initiation and study of the Menger covering property in the context of generalized topological spaces. Firstly, we defined in Definition 3.1 the notion of λ-Menger, studied some of its properties and provided several examples that illustrate some aspects of λ-Menger spaces. Secondly, we defined the notion of a λ-uniformly Menger space in Definition 4.2, gave a characterization of λ-uniformly Menger spaces in terms of λ-open covers, and examined its relationship with λ-Menger spaces.

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