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Some properties of *-weak rings with involution

Muna E. Abdulhafed[1*] and Aafaf E. Abduelhafid[2]

*Corresponding author: muna.am2016@gmail.com, Department of Mathematics, Faculty of Science, University of Azzaytuna, Tarhunah, Libya.

[2] Department of Mathematics, Faculty of Education, University of Azzaytuna, Tarhunah, Libya.

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Abstract
Throughout this paper, we introduced the concept of *-weak (*-IFP, quasi-*-IFP, and *-reversible) *-rings also study properties and the basic structure of *-weak *-rings, giving some of the results. Moreover, we will clarify the conditions for the *-weak *rings to extend into subrings with the involution of the ring, at the upper triangular matrices $(T_{n\times n}(R))$, with the same diagonal).

Keywords: *****-weak *-IFP; *****-weak quasi *-IFP; *****-weak *-reversible *-rings.

INTRODUCTION

All the rings, in this case, are associated with R unity. Mapping $*: R \longrightarrow R$ is an involution if $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in R$. A *-ring R is called *-zero divisor, for any $a \neq 0$. If $ab = ba^* = 0$, for some $0 \neq b \in R$ since R is an integral domain of involution if there are no nonzero zero divisors with involution, see [(Usama A. Aburawash and Khadija B. Sola, 2010)]. A self-adjoint idempotent; $e^2 = e = e^*$, is called projection. A nonempty subset S of a *-ring R is called self-adjoint or *-subset if $S^* = \{s^* | s \in S\} = S$, and from (U. A. Aburawash & Saad, 2016), defined semi-proper involution $*$ (resp. proper) if $aRa^* \neq 0$ (resp. $aa^* \neq 0$) for $0 \neq a \in R$, If $a^n = 0 = (aa^*)^m$ for some $n, m \in \mathbb{Z}^+$, then a is called *-nilpotent *-ring. Moreover, a *-ring R is said to be *-reduced if it has no nonzero *-nilpotent elements, and a *-ring R is called *-Baer if the *-right annihilator of each nonempty subset $A \in R$ is a principal *-bi-ideal generated by a projection; that is $r^*(A) = eRe$. Following (I. Kaplansky, 1968), A *-ring R is called Baer *-ring if the right annihilator of every nonempty subset of R is a principal right ideal generated by a projection, every Baer *-ring is a *-Baer ring with involution. From (Kim & Lee, 2003), R is semi-commutative or has an IFP ring if for every $a, b \in R$, if $ab = 0$ implies $aRb = 0$ or ($r(a)$ is an ideal of R for any $a \in R$. In (U. A. Aburawash & Saad, 2014) and (U. A. Aburawash & Saad, 2019), a *-ring R is called IFP with involution (resp. quasi-IFP with involution) if $ab = 0$ (resp. $ab = 0 = ab^*$) implies $aRb^* = 0$ (resp.; $aRb = 0$), for all $a, b \in R$. According to (U. A. Aburawash & Saad, 2019), (U. A. Aburawash & Abdulhafed, 2018b), and (U. A. Aburawash & Abdulhafed, 2018a), if $ab = 0 = ab^*$ implies $ba = 0$ (resp.; ba is central) for all $a, b \in R$ is called *-reversible (resp.; central *-reversible) *-ring. Every *-reversible *-ring has quasi-IFP with involution, a *-ring R is called weak *-IFP (resp.; weak quasi-IFP with involution), if $ab = 0$ (resp., $ab = 0 = ab^*$ for all $a, b \in R$ implies aRb^* is nilpotent (resp., aRb is nilpotent), a *-ring R central quasi IFP with involution (resp., has quasi-IFP with involution), if for all $a, b \in R$, $ab = 0 = ab^*$ implies aRb is central (resp., $aRb = 0$). the ring R is called weakly *-reversible, if $ab = ab^* = 0$, for all $a, b, r \in R$ then $Rbra$ is nil ideal of R.

In this article, every *-IFP is *-weak *-IFP, quasi *-IFP is *-weak quasi *-IFP and *-reversible is *-weak *-reversible, we study some properties *-weak *-rings it above. Moreover, a *-ring R is *-weak (*-IFP, quasi*-IFP, and *-reversible) if and only if for any n, the upper triangular matrix equal diagonal $T_{n\times n}(R)$ is *-weak (*-IFP, quasi*-IFP and *-reversible) *-ring. Further, we studied the extension of localization and Laurent polynomial of *-weak *-rings above.

***-weak *-IFP *-ring.**

Here section, we introduce another generalization for *-IFP; namely *-weak *-IFP *-rings.

Definition1. The *-ring R is called *- weak *-IFP, if $ab = 0$ implies aRb^* is *- nilpotent, for any $a, b \in R$.

Every *-IFP is *- weak *-IFP, and *- weak *-IFP is weak *-IFP. But the discourse is true by conduction semi-proper or central *- reversible.

Proposition1. If R is a weak *-IFP *-ring and semi-proper involution $*$ then R is *- weak *-IFP.

Proof. Let $ab = 0$ for some $a, b \in R$., $(arb^*)R(arb^*)^* = arb^*RbRa^* = a(rb^*R)bRa^* \subseteq aRbRa^* = 0$, by semi-proper involution $*$. Thus, R is *- weak *-IFP.

Proposition 2. Let R be a weak *-IFP *-ring and central *-reversible. Then R is *- weak *-IFP.

Proof. Let $ab = 0$ for some $a, b \in R$. Also, $bab = 0$ and $(baRb^*)^2 = baRb^*baRb^* = Rb^*babaRb^* = 0$,
$(baRb^*)(baRb^*)^* = baRb^*bRa^*b^* = Rb^*babRa^*b^* = 0$ for $r \in R$, by central *-reversible. Therefore, R is *- weak *-IFP.

Moreover, each commutative *-ring is *-weak *-IFP. The proof is easy with the following result.

Proposition 3. The class *-ring of the *-weak IFP with involution is closed (using changeless involution) by constructing *- subrings under its direct sums.

Proposition 4. Suppose that R be a commutative *-ring, then $T_{n\times n}(R)$ is a weak *-IFP *-ring, with involution * define as;

$$\begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-2)} & a_{2n} \\ 0 & 0 & a & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{n-1(n)} \\ 0 & 0 & 0 & \cdots & \cdots & a \end{pmatrix}^* = \begin{pmatrix} a & a_{(n-1)n} & a_{(n-2)n} & \cdots & a_{2n} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-2)} & a_{1(n-1)} \\ 0 & 0 & a & \cdots & \cdots & a_{1(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{12} \\ 0 & 0 & 0 & \cdots & \cdots & a \end{pmatrix}$$

(i.e., fix the two diagonals (right and left) with interchange the symmetric elements.)

Proof. If $A=(a_{ij})$ and $B=(b_{ij})\in T_{n\times n}(R)$ with $ab=0=ab^{*}$, where $1\le i\le j\le n$, then $ab=0$. By hypothesis, R is *-weak *-IFP, there exists $k\in\mathbb{N}$ such that $(acb)^{k}=0$ for any $C=(c_{ij})\in T_{n\times n}(R)$, since a, b and, c are the diagonal elements of A, B and, C with respective. Hence, there exists $n,m\in\mathbb{N}$ such that $((ACB^{*})^{k})^{n}=0,(((ACB^{*})(ACB^{*})^{*})^{k})^{m}=0$. Therefore, $T_{n\times n}(R)$ is *-weak *-IFP.

Now, every *-ring having*-IFP is *- weak*-IFP. The converse is true with semi-proper involution $*$, and each *- weak*-IFP *-ring is weak*-IFP *- ring while the converse is not true, as shown in the following example.

Example1. The *-ring $T_{3\times 3}(\mathbb{Z})$with the involution * given by: $\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix}^{*}=\begin{pmatrix} a & d & c\\ 0 & a & b\\ 0 & 0 & a\end{pmatrix}$ is *-weak *-IFP by Proposition 4. For $A=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ and $B=\begin{pmatrix} 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, we have $AB=0$ and $ARB^{*}=\begin{pmatrix} 0 & 0 & a\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\neq 0, a\in\mathbb{Z}$, so $T_{3\times 3}(\mathbb{Z})$ has not *-IFP.

There is a weak semi-commutative (IFP) *-ring which is not *-weak IFP with involution, see **[(U. A. Aburawash & Abdulhafed, 2018b)**, Example 9].
Next, by Proposition 4, the ring $T_{2\times 2}(R)$ is *-weak *-IFP. Also, $\mathbb{M}_{2\times 2}(R)$ with self- adjoint has not *-weak *-IFP, since $A=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}, B=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}$, satisfy $AB=0$ while $ACB^{*}=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} x & y\\ 0 & z\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix} x & 0\\ 0 & 0\end{pmatrix}\neq 0,\ \forall\, x,y,z\in R$.

Therefore, we note that by diagram in the class of *-rings.

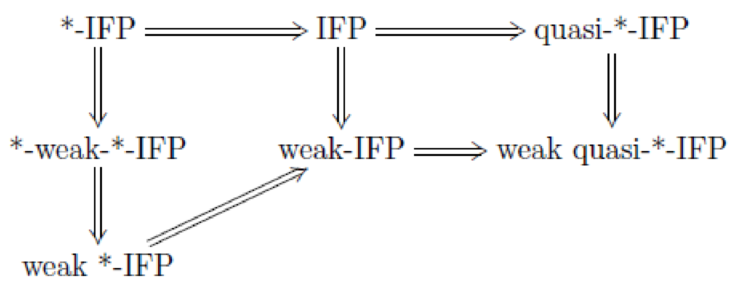


Diagram (1)

***- weak quasi *-IFP *-ring.**
In her section, we study the properties of the *-weak quasi-*-IFP rings with involution.

Definition 2. The *-ring R is called *- weak *-IFP, if for all $a,b\in R$, $ab=0=ab^{*}$ implies aRb is

*- nilpotent. Consequently, aRb^{*} is also *-nilpotent .Every commutative *-ring is *-weak quasi IFP with involution, each *-weak quasi-IFP with involution ring with involution is weak quasi-IFP with involution and *-weak quasi-IFP with involution ring with involution is quasi-IFP with involution, however, the converse is proper when the *-ring is semi-proper.

Proposition 5. Consider R is a weak quasi-*-IFP ring with involution and semi-proper involution $*$. Then R is the *- weak quasi-*-IFP *-ring.

Proof. Let $ab = 0 = ab^*$ for some $a, b \in R$, $(arb)R(arb)^* = arbRb^*Ra^* = a(rbR)b^*Ra^* \subseteq aRb^*Ra^* = 0$ for $a, b \in R$, by semi-proper involution $*$. Thus, R is *- weak *-IFP.

Proposition 6. Suppose that R is a central *-reversible *-ring and a weak quasi-IFP with involution. Then R is a *- weak quasi-IFP with involution.

Proof. If $ab = 0 = ab^*$ for some $a, b \in R$, then $bab = 0 = bab^*$ and $(baRb)^2 = baRbbaRb = baRbabRb = 0$,
$(baRb)(baRb)^* = baRbb^*Ra^*b^* = Rbbab^*Ra^*b^* = 0$, from central *-reversible. Thus, R is *-weak quasi-*-IFP.
Now, while there is no clear connection between *-weak quasi-*-IFP and weak- IFP. However, *-weak quasi-*-IFP R is weak- IFP if R has *-IFP.

Proposition 7. If R is *-weak quasi-*-IFP and R has IFP with involution, then R is a weak -IFP.

Proof. Since $ab = 0$ implies $aRb^* = 0$, by hypothesis and R is a weak IFP.

Further, the proposition below shows that *-subrings are the direct sums of the *-weak quasi-*-IFP *-ring.

Proposition 8. The class *-ring of the *-weak quasi-*-IFP is closed (using changeless involution) by constructing *- subrings under its direct sums.
Now, with the proof similar to Proposition 4, we get the following.

Proposition 9. It R is a commutative *-ring, $T_{n\times n}(R)$ is *-weak quasi-*-IFP, with involution * given in **Proposition 4.**
Next, is R is a commutative *-ring then

$$T_{n\times n}(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \middle| a, a_{ij} \in R, n \geq 3 \right\},$$

is *-weak quasi-*-IFP by **Proposition 9**. However, $T_{4\times 4}(R)$ is not quasi-*-IFP, so in general in case $n \geq 4$ is not quasi-*-IFP.
Moreover, as an instance, we note that the condition $T_{n\times n}(R)$of Proposition 9, cannot be weakened to the whole matrix *-ring $\mathbb{M}_{n\times n}(R)$, because n is more than 1.

Example 2. Let R be the *-ring of integer numbers ($\mathbb{Z}$). Consider the *-ring is *- weak quasi-IFP with involution, while the *-ring $\mathbb{M}_{2\times 2}(\mathbb{Z})$ with self-adjoint is not *-weak quasi-IFP with involution. For $A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -a \\ 0 & 0 \end{pmatrix}, a \in \mathbb{Z}$, satisfy $AB = 0 = AB^*$ and $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_{2\times 2}(R)$, thus $ACB = \begin{pmatrix} 0 & 0 \\ 0 & -a^2 \end{pmatrix}$ is not *-nilpotent.

Theorem1. Consider the following conditions, where R is a *-ring.

1. R is quasi-*-IFP.
2. R is central quasi-*-IFP.
3. R is *-weak quasi-*-IFP.
4. R is weak quasi-*-IFP.

Proof. $1 \Longrightarrow 2$: Obvious.
$2 \Longrightarrow 3$: : If $a, b \in R$ satisfied $ab = 0 = ab^*$, then arb, a^2rb, arb^2 and a^2rb^2 are central. Thus, $(arb)^2 = arbarb = a(arb)rb = r(a^2rb)b = ra^2rb^2 = ra(arb)b = r(arb)ab = 0$ and $((arb)(arb)^*)^2 = (arb)b^*r^*a^*(arb)b^*r^*a^* = b^*r^*a^*(arb)(arb)b^*r^*a^* =$ $b^*r^*a^*(aarbrb)b^*r^*a^* = b^*r^*a^*(a^2rbrb)b^*r^*a^* = b^*r^*a^*(ra^2rb^2)b^*r^*a^* =$ $b^*r^*a^*r(a(arb)b)b^*r^*a^* = b^*r^*a^*r((arb)ab)b^*r^*a^* = 0$
. Hence, arb is *-nilpotent for all $r \in R$ and R is *-weak quasi-*-IFP.
$3 \Longrightarrow 4$: It's clear.
Moreover, the discussion of Theorem 1 is invalidated by [(U. A. Aburawash & Abdulhafed, 2018a), Proposition 1] and Propositions (5 and 6).
The outcome can be obtained from [(U. A. Aburawash & Abdulhafed, 2018a), Proposition 1]and Theorem 1.

Corollary 1. Let R be *-ring and central quasi-IFP with involution, if R is a Baer ring with involution, then R is *-weak quasi-IFP with involution.

We know, every quasi-IFP with involution is *-weak quasi-IFP with involution, and by [(M. S. and U. A. Aburawash, 2023), propositions (4.2 and 5.1) and theorem 5.6)] the conclusion is obtained.

Corollary 2. Let R be a ring with involution and I a proper ideal a with involution of R. If I is a reduced with involution ring with involution (without identity) and R/I has quasi-IFP with involution, then R is also *-weak quasi-IFP with involution.

Corollary 3. Let R be a reduced with involution, ring with involution, and I an ideal with involution of R with IFP (as a ring without identity). If R has quasi-IFP with involution, then the *-subring S of the upper triangular matrix ring $T_{3\times 3}(R)$ over R is defined as follows:

$$S = \left\{ \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in I \right\}, \text{ is *-weak quasi-*-IFP}$$

Corollary 4. Let R be a *-reduced *-ring and n any positive integer. If R is *-weak quasi-$*$-IFP, then $R[x]/\langle x^n \rangle$ is *-weak quasi-$*$-IFP.
Thus, we note that by a diagram, of *-rings.

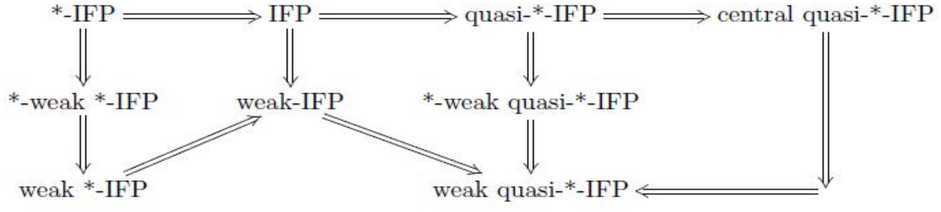


Diagram (2)

***- weak *-reversible *-ring.**
Here section, we introduce another generalization for *-reversible; namely *-weakly *-reversible *-rings.

Definition 3: A *-ring R is called *-weak *-reversible if for all $a, b, r \in R$, $ab = 0 = ab^*$, implies $Rbra$ is a *-nil left (equivalently, $braR$ is a *-nil right) *-ideal of R. Consequently, Rb^*ra is a *-nil left (equivalently, b^*raR is a *-nil right) *-ideal of R.

The commutative *-ring is *-weak *-reversible and each *-weak *-reversible *-ring is weak *-reversible. The converse is true while the *- ring has *-IFP as proven inside the following.

Proposition 10: If R *- ring has *-IFP and *-weak *-reversible, then R is weakly reversible**.**
Now, each *-weak *-reversible *-ring is weak *-reversible. The converse is true while the *- ring has semi-proper involution $*$ and central *-reflexive, we have the subsequent proposition.

Proposition 11: If R *-ring has semi-proper involution $*$ and weak *-reversible, then R is *-weak *-reversible.

Proof. Let $ab = 0 = ab^*$ for some $a, b \in R$. Then, $(Rbra)R(Rbra)^* = Rbra(Ra^*r^*)b^*R \subseteq RbraRb^*R = 0$, for all $r \in R$, from semi-proper involution $*$. Hence, R is *-weak *-reversible.

Recall that, [(U. A. Aburawash & Saad, 2019) and (U. A. Aburawash & Abdulhafed, 2018a)], if $ab = ab^* = 0$ implies $bRa = 0$ (resp., bRa is central) for all $a, b \in R$ is called *-reflexive (resp., central *-reflexive) *-ring. Every *-reversible is *-reflexive ring with involution.

Proposition 12: If R is weak *-reversible and central *-reflexive, then R is *-weak *-reversible since R is a *-ring.

Proof. Let $ab = 0 = ab^*$ for some $a, b \in R$. Then $(Rbra)^2 = RbraRbra = RRbrabra = 0$, $(Rbra)(Rbra)^* = Rbraa^*r^*b^*R = Ra^*r^*brab^*R = 0$, for all $r \in R$, from central *-reflexive. Hence, R is *- weak *-reversible.
Now, prove the next result it is easy.

Proposition 13. The *-weak *-reversible ring with involution class is closed under taking subrings with involution and under direct sums (with changeless involution).

Now, the next proposition is similar to **Proposition 4.**

Proposition 14. R is a commutative *-ring, $T_{n\times n}(R)$ is *- weak *-reversible, with involution * from **Proposition 4.**

In example 6 of **(U. A. Aburawash & Abdulhafed, 2018b)** there exists a *-weak *-reversible and quasi *-IFP *-ring which is not *-reversible.

Next, for example, we note that the condition $T_{n\times n}(R)$ of Proposition 14, cannot be weakened to the whole matrix *-ring $\mathbb{M}_{n\times n}(R)$, since n is more than 1.

Example 3. If R is a *-weak *-reversible *-ring and n is more than 1, then $\mathbb{M}_{2\times 2}(R)$, with adjoint involution, is not *-weak *-reversible. For $X = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,
we have $XY = 0 = XY^*$ and for $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_{2\times 2}(R)$, we see that $RYZX = \begin{pmatrix} 0 & 2a \\ 0 & 2c \end{pmatrix}$ is not *-nil, $\forall\, a, c \in R$.

The subsequent result shows that central *-reversible*-rings lie properly between *-reversible and *-weak*-reversible rings with involution.

Theorem 2. Consider the following conditions, where R is a *-ring.

1. R is *-reversible.
2. R is central *-reversible.
3. R is *-weak *-reversible.
4. R is weak *-reversible.

Proof. $1 \Longrightarrow 2$: It's clear.
$2 \Longrightarrow 3$: If $ab = 0 = ab^*$, then $r_1ab = 0 = r_1ab^*$ for all $a, b, r_1 \in R$, and br_1a is central, since R is central *-reversible. Hence, $(rbr_1a)^2 = rbr_1arbr_1a = rrbr_1abr_1a = 0$ and $(rbr_1a)(rbr_1a)^* = rbr_1aa^*r_1{}^*b^*r^* = ra^*r_1{}^*br_1ab^*r^* = 0$ for all $r \in R$ and R is weak *-reversible.
$3 \Longrightarrow 4$: It's clear.
By [(U. A. Aburawash & Abdulhafed, 2018b), Propositions 3, 4] and Propositions (11 and 12) the converse of Theorem 2 isn't true.
Further, we get corollaries from [(U. A. Aburawash & Abdulhafed, 2018b), Proposition 4] and Theorem 2.

Corollary 5. Every *-domain is a *-weak *-reversible*-ring.

Corollary 6. If R is a central *-reversible and *-Baer ring with involution, then R is *-weak *-reversible.
It's important to remember that every Bear *-ring is a *-Bear *-ring. We have the following result.

Corollary 7. If R is a Baer and central *-reversible *-ring, then R is *-weak *-reversible.
Now, as shown by the next theorem, the central *-reversible*-rings properly lie between the classes of *-reversible, weak quasi-*-IFP, and *-weak quasi-*-IFP*-rings

Theorem 3. Consider the following conditions, where R is a *-ring.

1. R is *-reversible.
2. R is central *-reversible.
3. R is weak quasi-*-IFP.
4. R is *-weak quasi-*-IFP.

Proof. $1 \Longrightarrow 2$: Obviously.

$2 \Longrightarrow 3$: By [(U. A. Aburawash & Abdulhafed, 2018b), **Theorem 2**].
$3 \Longrightarrow 4$: From Propositions (5 and 6).
The converse of **Theorem 3** is not true from [(U. A. Aburawash & Abdulhafed, 2018b), **Example 6**].

The following results, from [(U. A. Aburawash & Saad, 2019), **Proposition 10**], [(U. A. Aburawash & Abdulhafed, 2018b), **Corollaries** (**7, 8, 9, 10**)], and **Theorem 2.**

Corollary 8. If R is *-ring and central reduced then $T(R;R)$ is a *-weak *-reversible.

Corollary 9. If the *-ring R is reduced then $T(R;R)$ is a *-weak *-reversible.

Corollary 10. If the *-ring R is *-reduced and *-reversible, then $T(R;R)$, with componentwise involution, is *-weak *-reversible.

Corollary 11. If the *-ring R is reduced and *-reversible, then $T(R;R)$, with component wise involution, is *-weak *-reversible.
Thus, we note that by the diagram, of *-rings.

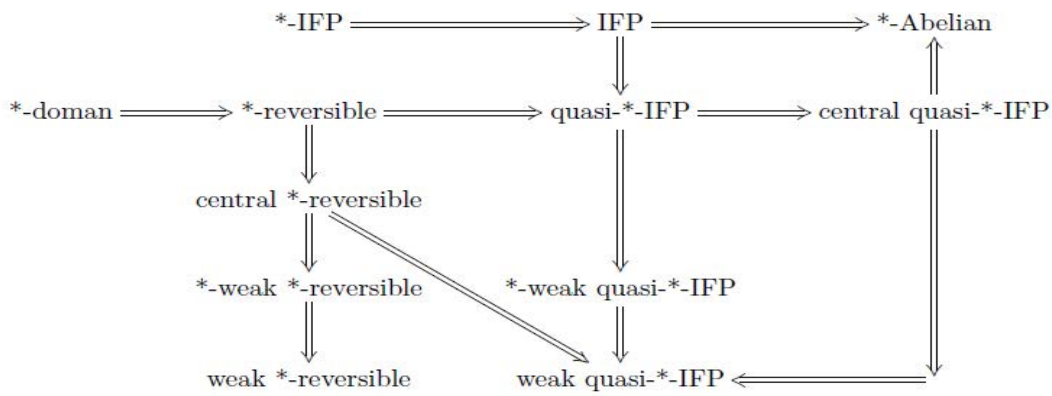


Diagram (3)

Extending of *-weak (*-IFP, quasi *-IFP, and *-reversible) rings with involution.

We will here focus on the properties of *-weak (*-IFP, the quasi –IFP with involution, and *-reversible) rings with involution is proven to be extended from a *-ring to its localization and Laurent polynomial, the Dorroh extension, and from Ore *-ring to its classical quotient.

If $\Omega^{-1}R = \{u^{-1}a \mid u \in \Omega, a \in R\}$, then $\Omega^{-1}R$ is a *-ring with involution $\diamond$ defined as $(u^{-1}a)^{\diamond} = u^{-1*}a^{*} = u^{*-1}a^{*}$ where R is a ring with involution, Ω is a multiplicative in R consisting of R and central regular elements, see **[(U. A. Aburawash & Abdulhafed, 2018b)].**
For it, we have the propositions.

Proposition 15. If e is a central projection of R since R is a *-ring, then it's equivalent.

1. R is *-weak quasi-*-IFP.
2. eR and $(1-e)R$ are *-weak quasi-*-IFP.
3. $\Omega^{-1}R$ is *-weak quasi-*-IFP.

Proof. $1 \Leftrightarrow 2$: It is clear that this is straightforward since subrings with involution and finite direct products of *-weak quasi-*-IFP ring with involution is *-weak quasi-*-IFP.
$3 \Longrightarrow 1$: It is clear since R is a subring with involution of $\Omega^{-1}R$.

$1 \Longrightarrow 3$: Since R is a *-subring of $\Omega^{-1}R$. Let $\alpha\beta = 0 = \alpha\beta^{\circ}$with $\alpha = u^{-1}a, \beta = v^{-1}b, u, v \in \Omega$ and a, b ∈ R, and let $\gamma = w^{-1}c$ for any element of $\Omega^{-1}R, w \in \Omega, c \in$ R. Since Ω contained in the center of R, we have $0 = \alpha\beta = u^{-1}av^{-1}b = (u^{-1}v^{-1})ab$, and $0 = \alpha\beta^{\circ} = u^{-1}a(v^{-1}b)^{\circ} = (u^{-1}(v^{-1})^{*})ab^{*}$, and hence ab=0=ab^*, but R *-weak quasi-*-IFP, like this for some n and m in positive integers that $(\text{acb})^{n} = 0, ((\text{acb})(\text{acb})^{*})^{m} = 0$. Thus $(\alpha\gamma\beta)^{n} = (u^{-1}aw^{-1}cv^{-1}b)^{n} = ((vwa)^{-1}acb)^{n} = ((vwa)^{-1})^{n}(acb)^{n} = 0$ and $((\alpha\gamma\beta)(\alpha\gamma\beta)^{\circ})^{n} = ((u^{-1}aw^{-1}cv^{-1}b)(\beta^{\circ}\gamma^{\circ}\alpha^{\circ}))^{n} = ((u^{-1}aw^{-1}cv^{-1}b)(\beta^{\circ}\gamma^{\circ}\alpha^{\circ}))^{m} = ((vwa)^{-1}(acb)(b^{*}(v^{-1})^{*}c^{*}(w^{-1})^{*}a^{*}(u^{-1})^{-1})^{m} = ((vwa)^{-1}(acb)(u^{*}w^{*}v^{*})^{-1}(b^{*}c^{*}a^{*}))^{m} = ((vwa)^{-1}(u^{*}w^{*}v^{*})^{-1}(acb)(b^{*}c^{*}a^{*}))^{m} = ((vwa)^{-1}(u^{*}w^{*}v^{*})^{-1})^{m}((acb)(acb)^{*}))^{m}$
. Therefore, $\Omega^{-1}R$ is *-weak quasi-*-IFP.

Proposition 16. If R is a *-ring, and e has a central projection in R, the following is its equivalent:

1. R is *-weak *-reversible.
2. eR and $(1-e)R$ are *-weak *-reversible.
3. $\Omega^{-1}R$ is *-weak *-reversible.

Proof. It is similar to that of **Proposition 15.**
We get results from**[(U. A. Aburawash & Abdulhafed, 2018b)**, Proposition 6].

Corollary 12. Let R be a *-ring, and e is a central projection of R. Then eR and $(1-e)R$ are *-reversible if R is *- weak *-reversible.

Corollary 13. If R is *-reversible a *-ring and e central of $e^2 = e = e^*$in R. Then eR and $(1-e)R$ are *-weak *-reversible.

Then, the ring with involution of the Laurent polynomials in x, with coefficients in the *-ring R, consists of all formal sums $f(x) = \sum_{i=k}^{n} a_i x^i$ an involution *, $f^*(x) = \sum_{i=k}^{n} a_i^* x^i$ with explicit addition and multiplication, where $a_i \in R$ and k, n (possibly negative) integers. We denote this plate by $R[x; x^{-1}]$.

Corollary 14. For a *-ring R, Ω is *-weak quasi-*-IFP iff $R[x; x^{-1}]$ is *-weak quasi-IFP with involution.

Proof. It is sufficient to establish the necessity since $R[x]$ is a subring with involution of $R[x; x^{-1}]$. Let $\Omega = \{1, x, x^2, x^3, x^4, \cdots\}$. Then Ω is a multiplicative closed subset with the involution of Ω. Where $R[x; x^{-1}] = \Omega^{-1}R[x]$, it follows that $R[x; x^{-1}]$ is *-weak quasi-*-IFP by Proposition 15.

Corollary 15. For R *-ring, Ω is *-weak *-reversible if and if only $R[x; x^{-1}]$ is *-weak *-reversible.

Proof. Like **Corollary** 14 using Proposition 16.
Accordingly, we have the equivalence on *-weak quasi-*-IFP and *-weak *-reversibility of another situation.

Corollary 16. If R is a *-ring the following is its equivalent:

1) R is *-weak quasi-IFP with involution.
2) $R[x]$ is *-weak quasi-IFP with involution.
3) $R[x; x^{-1}]$ is *-weak quasi-IFP with involution.

Corollary 17. For a *-ring R, the following is its equivalent:

1. R is *-weak *-reversible.
2. $R[x]$ is *-weak *-reversible.
3. $R[x;x^{-1}]$ is *-weak *-reversible.

Then, for the Dorroh expansion of the *-ring $\mathrm{D(R,\mathbb{Z}) = \{(r,n): r \in R, n \in \mathbb{Z}\}}$ R is a ring whose functions are $(r_1,n_1)+(r_2,n_2)=(r_1+r_2,n_1+n_2)$ and $(r_1,n_1)(r_2,n_2)=(r_1r_2+n_1r_2+n_2r_1,n_1n_2)$. The R involution can naturally extend to $D(R,\mathbb{Z})$ in the form $(r,n)^*=(r^*,n)$ (see (U. A. Aburawash, 1997).

Next, we get the following results by [(U. A. Aburawash & Abdulhafed, 2018b), **Proposition 21**]:

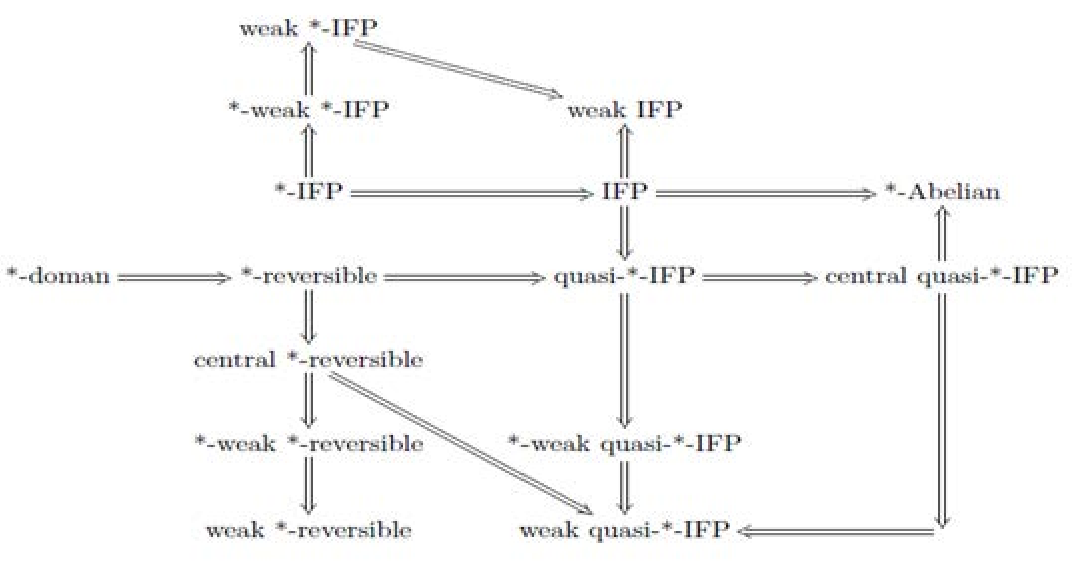
Corollary 18. A *-ring R is *-reversible, and the Dorroh extension $D(R,\mathbb{Z})$ of R is *-weak-*-reversible.

Corollary 19. A *-ring R is Dorroh extension $D(R,\mathbb{Z})$ is *-reversible, R is *-weak-*-reversible.
We know that Let R be a ring with involution * which is a left order in a ring Q. Then R is a right order in Q and Q has an involution given by$(a^{-1}b)^*=b^*(a^*)^{-1}$ (see [(Martindale & 3rd, 1969), Lemma 4)], and [(U. A. Aburawash & Abdulhafed, 2018b), Theorem 4] given the following results.

Corollary 20. If R is *-reversible, then Q is *-weak-*-reversible.

Corollary 21. If Q is *-reversible, then R is *-weak-*-reversible.

CONCLUSION



Here, we conclude the paper's results using diagrams to explain the relations among the corresponding classes, from diagrams [(1), (2)] and diagram (3).

Furthermore, we have the following conclusions:
Every *-IFP (resp., quasi *-IFP and *- reversible) are *-weak *-IFP (resp., *-weak quasi *-IFP and *-weak-*-reversible) rings with involution, also, each *-IFP is *-weak quasi *-IFP ring with involution. Moreover, every *-Doman is *-weak *-reversible ring with involution.

Finally, we future studies about properties of *-weak *- rings of (*-reflexive, S. *-reversible, and S. *-reflexive) rings with involution.

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