

## Some properties of \*-weak rings with involution

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## Received:

08 July 2024

## Accepted:

29 August 2024

## Publish online:

31 August 2024

## Abstract

Throughout this paper, we introduced the concept of \*-weak (\*-IFP, quasi-\*IFP, and \*-reversible) \*-rings also study properties and the basic structure of \*-weak \*-rings, giving some of the results. Moreover, we will clarify the conditions for the \*-weak \*-rings to extend into subrings with the involution of the ring, at the upper triangular matrices ( $T_{n \times n}(R)$ , with the same diagonal).

**Keywords:** \*-weak \*-IFP; \*-weak quasi \*-IFP; \*-weak \*-reversible \*-rings.

## INTRODUCTION

All the rings, in this case, are associated with  $R$  unity. Mapping  $*: R \rightarrow R$  is an involution if  $(a + b)^* = a^* + b^*$ ,  $(ab)^* = b^*a^*$  and  $(a^*)^* = a$  for all  $a, b \in R$ . A \*-ring  $R$  is called \*-zero divisor, for any  $a \neq 0$ . If  $ab = ba^* = 0$ , for some  $0 \neq b \in R$  since  $R$  is an integral domain of involution if there are no nonzero zero divisors with involution, see [(Usama A. Aburawash and Khadija B. Sola, 2010)]. A self-adjoint idempotent;  $e^2 = e = e^*$ , is called projection. A nonempty subset  $S$  of a \*-ring  $R$  is called self-adjoint or \*-subset if  $S^* = \{s^* | s \in S\} = S$ , and from (U. A. Aburawash & Saad, 2016), defined semi-proper involution \* (resp. proper) if  $aRa^* \neq 0$  (resp.  $aa^* \neq 0$ ) for  $0 \neq a \in R$ . If  $a^n = 0 = (aa^*)^m$  for some  $n, m \in \mathbb{Z}^+$ , then  $a$  is called \*-nilpotent \*-ring. Moreover, a \*-ring  $R$  is said to be \*-reduced if it has no nonzero \*-nilpotent elements, and a \*-ring  $R$  is called \*-Baer if the \*-right annihilator of each nonempty subset  $A \in R$  is a principal \*-bi-ideal generated by a projection; that is  $r^*(A) = eRe$ . Following (I. Kaplansky, 1968), A \*-ring  $R$  is called Baer \*-ring if the right annihilator of every nonempty subset of  $R$  is a principal right ideal generated by a projection, every Baer \*-ring is a \*-Baer ring with involution. From (Kim & Lee, 2003),  $R$  is semi-commutative or has an IFP ring if for every  $a, b \in R$ , if  $ab = 0$  implies  $aRb = 0$  or ( $r(a)$  is an ideal of  $R$  for any  $a \in R$ ). In (U. A. Aburawash & Saad, 2014) and (U. A. Aburawash & Saad, 2019), a \*-ring  $R$  is called IFP with involution (resp. quasi-IFP with involution) if  $ab = 0$  (resp.  $ab = 0 = ab^*$ ) implies  $aRb^* = 0$  (resp.;  $aRb = 0$ ), for all  $a, b \in R$ . According to (U. A. Aburawash & Saad, 2019), (U. A. Aburawash & Abdulhafed, 2018b), and (U. A. Aburawash & Abdulhafed, 2018a), if  $ab = 0 = ab^*$  implies  $ba = 0$  (resp.;  $ba$  is central) for all  $a, b \in R$  is called \*-reversible (resp.; central \*-reversible) \*-ring. Every \*-reversible \*-ring has quasi-IFP with involu-



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tion, a  $*$ -ring  $R$  is called weak  $*$ -IFP (resp.; weak quasi-IFP with involution), if  $ab = 0$  (resp.,  $ab = 0 = ab^*$  for all  $a, b \in R$ ) implies  $aRb^*$  is nilpotent (resp.,  $aRb$  is nilpotent), a  $*$ -ring  $R$  central quasi IFP with involution (resp., has quasi-IFP with involution), if for all  $a, b \in R$ ,  $ab = 0 = ab^*$  implies  $aRb$  is central (resp.,  $aRb = 0$ ). the ring  $R$  is called weakly  $*$ -reversible, if  $ab = ab^* = 0$ , for all  $a, b, r \in R$  then  $Rbra$  is nil ideal of  $R$ .

In this article, every  $*$ -IFP is  $*$ -weak  $*$ -IFP, quasi  $*$ -IFP is  $*$ -weak quasi  $*$ -IFP and  $*$ -reversible is  $*$ -weak  $*$ -reversible, we study some properties  $*$ -weak  $*$ -rings it above. Moreover, a  $*$ -ring  $R$  is  $*$ -weak ( $*$ -IFP, quasi $*$ -IFP, and  $*$ -reversible) if and only if for any  $n$ , the upper triangular matrix equal diagonal  $T_{n \times n}(R)$  is  $*$ -weak ( $*$ -IFP, quasi $*$ -IFP and  $*$ -reversible)  $*$ -ring. Further, we studied the extension of localization and Laurent polynomial of  $*$ -weak  $*$ -rings above.

### $*$ -weak $*$ -IFP $*$ -ring.

Here section, we introduce another generalization for  $*$ -IFP; namely  $*$ -weak  $*$ -IFP  $*$ -rings.

**Definition1.** The  $*$ -ring  $R$  is called  $*$ - weak  $*$ -IFP, if  $ab = 0$  implies  $aRb^*$  is  $*$ - nilpotent, for any  $a, b \in R$ .

Every  $*$ -IFP is  $*$ - weak  $*$ -IFP, and  $*$ - weak  $*$ -IFP is weak  $*$ -IFP. But the discourse is true by conduction semi-proper or central  $*$ - reversible.

**Proposition1.** If  $R$  is a weak  $*$ -IFP  $*$ -ring and semi-proper involution  $*$  then  $R$  is  $*$ - weak  $*$ -IFP.

**Proof.** Let  $ab = 0$  for some  $a, b \in R$ ,  $(arb^*)R(arb^*)^* = arb^*RbRa^* = a(rb^*R)bRa^* \subseteq aRbRa^* = 0$ , by semi-proper involution  $*$ . Thus,  $R$  is  $*$ - weak  $*$ -IFP.

**Proposition 2.** Let  $R$  be a weak  $*$ -IFP  $*$ -ring and central  $*$ -reversible. Then  $R$  is  $*$ - weak  $*$ -IFP.

**Proof.** Let  $ab = 0$  for some  $a, b \in R$ . Also,  $bab = 0$  and  $(baRb^*)^2 = baRb^*baRb^* = Rb^*babRb^* = 0$ ,  $(baRb^*)(baRb^*)^* = baRb^*bRa^*b^* = Rb^*babRa^*b^* = 0$  for  $r \in R$ , by central  $*$ -reversible. Therefore,  $R$  is  $*$ - weak  $*$ -IFP.

Moreover, each commutative  $*$ -ring is  $*$ -weak  $*$ -IFP. The proof is easy with the following result.

**Proposition 3.** The class  $*$ -ring of the  $*$ -weak IFP with involution is closed (using changeless involution) by constructing  $*$ - subrings under its direct sums.

**Proposition 4.** Suppose that  $R$  be a commutative  $*$ -ring, then  $T_{n \times n}(R)$  is a weak  $*$ -IFP  $*$ -ring, with involution  $*$  define as;

$$\begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-2)} & a_{2n} \\ 0 & 0 & a & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{n-1(n)} \\ 0 & 0 & 0 & \cdots & \cdots & a \end{pmatrix}^* = \begin{pmatrix} a & a_{(n-1)n} & a_{(n-2)n} & \cdots & a_{2n} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-2)} & a_{1(n-1)} \\ 0 & 0 & a & \cdots & \cdots & a_{1(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_{12} \\ 0 & 0 & 0 & \cdots & \cdots & a \end{pmatrix}$$

(i.e., fix the two diagonals (right and left) with interchange the symmetric elements.)

**Proof.** If  $A = (a_{ij})$  and  $B = (b_{ij}) \in T_{n \times n}(R)$  with  $ab = 0 = ab^*$ , where  $1 \leq i \leq j \leq n$ , then  $ab = 0$ . By hypothesis,  $R$  is  $*$ -weak  $*$ -IFP, there exists  $k \in \mathbb{N}$  such that  $(acb)^k = 0$  for any  $C = (c_{ij}) \in T_{n \times n}(R)$ , since  $a, b$  and,  $c$  are the diagonal elements of  $A, B$  and,  $C$  with respective. Hence, there exists  $n, m \in \mathbb{N}$  such that  $((ACB^*)^k)^n = 0, (((ACB^*)(ACB^*)^*)^k)^m = 0$ . Therefore,  $T_{n \times n}(R)$  is  $*$ -weak  $*$ -IFP.

Now, every  $*$ -ring having  $*$ -IFP is  $*$ -weak  $*$ -IFP. The converse is true with semi-proper involution  $*$ , and each  $*$ -weak  $*$ -IFP  $*$ -ring is weak  $*$ -IFP  $*$ -ring while the converse is not true, as shown in the following example.

**Example 1.** The  $*$ -ring  $T_{3 \times 3}(\mathbb{Z})$  with the involution  $*$  given by:  $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$  is  $*$ -weak  $*$ -IFP by Proposition 4. For  $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , we have  $AB = 0$  and  $ARB^* = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0, a \in \mathbb{Z}$ , so  $T_{3 \times 3}(\mathbb{Z})$  has not  $*$ -IFP.

There is a weak semi-commutative (IFP)  $*$ -ring which is not  $*$ -weak IFP with involution, see [(U. A. Aburawash & Abdulhafed, 2018b), Example 9].

Next, by Proposition 4, the ring  $T_{2 \times 2}(R)$  is  $*$ -weak  $*$ -IFP. Also,  $M_{2 \times 2}(R)$  with self-adjoint has not  $*$ -weak  $*$ -IFP, since  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , satisfy  $AB = 0$  while  $ACB^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \neq 0, \forall x, y, z \in R$ .

Therefore, we note that by diagram in the class of  $*$ -rings.

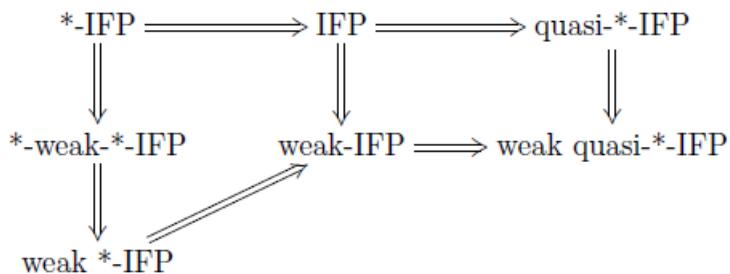


Diagram (1)

### $*$ -weak quasi $*$ -IFP $*$ -ring.

In her section, we study the properties of the  $*$ -weak quasi  $*$ -IFP rings with involution.

**Definition 2.** The  $*$ -ring  $R$  is called  $*$ -weak  $*$ -IFP, if for all  $a, b \in R, ab = 0 = ab^*$  implies  $aRb$  is

$*$ -nilpotent. Consequently,  $aRb^*$  is also  $*$ -nilpotent. Every commutative  $*$ -ring is  $*$ -weak quasi IFP with involution, each  $*$ -weak quasi-IFP with involution ring with involution is weak quasi-IFP with involution and  $*$ -weak quasi-IFP with involution ring with involution is quasi-IFP with involution, however, the converse is proper when the  $*$ -ring is semi-proper.

**Proposition 5.** Consider  $R$  is a weak quasi-\* -IFP ring with involution and semi-proper involution \*. Then  $R$  is the \*- weak quasi-\* -IFP \*-ring.

**Proof.** Let  $ab = 0 = ab^*$  for some  $a, b \in R$ ,  $(arb)R(arb)^* = arbRb^*Ra^* = a(rbR)b^*Ra^* \subseteq aRb^*Ra^* = 0$  for  $a, b \in R$ , by semi-proper involution \*. Thus,  $R$  is \*- weak \*-IFP.

**Proposition 6.** Suppose that  $R$  is a central \*-reversible \*-ring and a weak quasi-IFP with involution. Then  $R$  is a \*- weak quasi-IFP with involution.

**Proof.** If  $ab = 0 = ab^*$  for some  $a, b \in R$ , then  $bab = 0 = bab^*$  and  $(baRb)^2 = baRbbbaRb = baRbabRb = 0$ ,  $(baRb)(baRb)^* = baRbb^*Ra^*b^* = Rbbab^*Ra^*b^* = 0$ , from central \*-reversible. Thus,  $R$  is \*- weak quasi-\* -IFP.

Now, while there is no clear connection between \*-weak quasi-\* -IFP and weak- IFP. However, \*- weak quasi-\* -IFP  $R$  is weak- IFP if  $R$  has \*-IFP.

**Proposition 7.** If  $R$  is \*-weak quasi-\* -IFP and  $R$  has IFP with involution, then  $R$  is a weak -IFP.

**Proof.** Since  $ab = 0$  implies  $aRb^* = 0$ , by hypothesis and  $R$  is a weak IFP.

Further, the proposition below shows that \*-subrings are the direct sums of the \*-weak quasi-\* -IFP \*-ring.

**Proposition 8.** The class \*-ring of the \*-weak quasi-\* -IFP is closed (using changeless involution) by constructing \*- subrings under its direct sums.

Now, with the proof similar to Proposition 4, we get the following.

**Proposition 9.** It  $R$  is a commutative \*-ring,  $T_{n \times n}(R)$  is \*-weak quasi-\* -IFP, with involution \* given in **Proposition 4**.

Next, is  $R$  is a commutative \*-ring then

$$T_{n \times n}(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \middle| a, a_{ij} \in R, n \geq 3 \right\},$$

is \*-weak quasi-\* -IFP by **Proposition 9**. However,  $T_{4 \times 4}(R)$  is not quasi-\* -IFP, so in general in case  $n \geq 4$  is not quasi-\* -IFP.

Moreover, as an instance, we note that the condition  $T_{n \times n}(R)$  of Proposition 9, cannot be weakened to the whole matrix \*-ring  $M_{n \times n}(R)$ , because  $n$  is more than 1.

**Example 2.** Let  $R$  be the \*-ring of integer numbers ( $\mathbb{Z}$ ). Consider the \*-ring is \*- weak quasi-IFP with involution, while the \*-ring  $M_{2 \times 2}(\mathbb{Z})$  with self-adjoint is not \*-weak quasi-IFP with involution. For  $A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & -a \\ 0 & 0 \end{pmatrix}$ ,  $a \in \mathbb{Z}$ , satisfy  $AB = 0 = AB^*$  and  $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_{2 \times 2}(R)$ , thus  $ACB = \begin{pmatrix} 0 & 0 \\ 0 & -a^2 \end{pmatrix}$  is not \*-nilpotent.

**Theorem1.** Consider the following conditions, where  $R$  is a \*-ring.

1.  $\mathbf{R}$  is quasi-\*-IFP.
2.  $\mathbf{R}$  is central quasi-\*-IFP.
3.  $\mathbf{R}$  is \*-weak quasi-\*-IFP.
4.  $\mathbf{R}$  is weak quasi-\*-IFP.

**Proof. 1  $\Rightarrow$  2:** Obvious.

**2  $\Rightarrow$  3:** If  $a, b \in R$  satisfied  $ab = 0 = ab^*$ , then  $arb, a^2rb, arb^2$  and  $a^2rb^2$  are central. Thus,  $(arb)^2 = arbarb = a(arb)rb = r(a^2rb)b = ra^2rb^2 = ra(arb)b = r(arb)ab = 0$  and  $((arb)(arb)^*)^2 = (arb)b^*r^*a^*(arb)b^*r^*a^* = b^*r^*a^*(arb)(arb)b^*r^*a^* = b^*r^*a^*(aarbrb)b^*r^*a^* = b^*r^*a^*(a^2rbrb)b^*r^*a^* = b^*r^*a^*(ra^2rb^2)b^*r^*a^* = b^*r^*a^*r(arb)ab = b^*r^*a^*r((arb)ab)b^*r^*a^* = 0$ .

Hence,  $arb$  is \*-nilpotent for all  $r \in R$  and  $\mathbf{R}$  is \*-weak quasi-\*-IFP.

**3  $\Rightarrow$  4:** It's clear.

Moreover, the discussion of Theorem 1 is invalidated by [(U. A. Aburawash & Abdulhafed, 2018a), Proposition 1] and Propositions (5 and 6).

The outcome can be obtained from [(U. A. Aburawash & Abdulhafed, 2018a), Proposition 1] and Theorem 1.

**Corollary 1.** Let  $\mathbf{R}$  be \*-ring and central quasi-IFP with involution, if  $\mathbf{R}$  is a Baer ring with involution, then  $\mathbf{R}$  is \*-weak quasi-IFP with involution.

We know, every quasi-IFP with involution is \*-weak quasi-IFP with involution, and by [(M. S. and U. A. Aburawash, 2023), propositions (4.2 and 5.1) and theorem 5.6)] the conclusion is obtained.

**Corollary 2.** Let  $\mathbf{R}$  be a ring with involution and  $I$  a proper ideal with involution of  $\mathbf{R}$ . If  $I$  is a reduced with involution ring with involution (without identity) and  $\mathbf{R}/I$  has quasi-IFP with involution, then  $\mathbf{R}$  is also \*-weak quasi-IFP with involution.

**Corollary 3.** Let  $\mathbf{R}$  be a reduced with involution, ring with involution, and  $I$  an ideal with involution of  $\mathbf{R}$  with IFP (as a ring without identity). If  $\mathbf{R}$  has quasi-IFP with involution, then the \*-subring  $S$  of the upper triangular matrix ring  $T_{3 \times 3}(\mathbf{R})$  over  $\mathbf{R}$  is defined as follows:

$$S = \left\{ \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in I \right\}, \text{ is *-weak quasi-*-IFP}$$

**Corollary 4.** Let  $\mathbf{R}$  be a \*-reduced \*-ring and  $n$  any positive integer. If  $\mathbf{R}$  is \*-weak quasi-\*IFP, then  $\mathbf{R}[x]/\langle x^n \rangle$  is \*-weak quasi-\*IFP.

Thus, we note that by a diagram, of \*-rings.

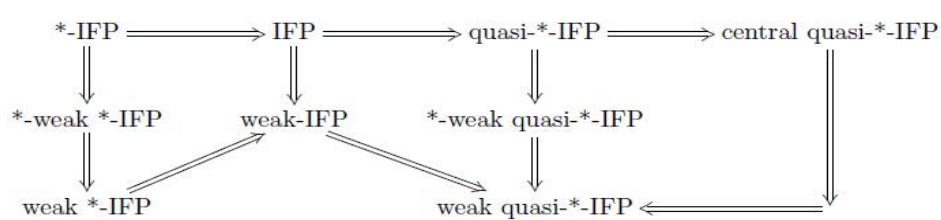


Diagram (2)

### \*- weak \*-reversible \*-ring.

Here section, we introduce another generalization for \*-reversible; namely \*-weakly \*-reversible \*-rings.

**Definition 3:** A \*-ring  $R$  is called \*-weak \*-reversible if for all  $a, b, r \in R$ ,  $ab = 0 = ab^*$ , implies  $Rbra$  is a \*-nil left (equivalently,  $braR$  is a \*-nil right) \*-ideal of  $R$ . Consequently,  $Rb^*ra$  is a \*-nil left (equivalently,  $b^*raR$  is a \*-nil right) \*-ideal of  $R$ .

The commutative \*-ring is \*-weak \*-reversible and each \*-weak \*-reversible \*-ring is weak \*-reversible. The converse is true while the \*-ring has \*-IFP as proven inside the following.

**Proposition 10:** If  $R$  \*-ring has \*-IFP and \*-weak \*-reversible, then  $R$  is weakly reversible.

Now, each \*-weak \*-reversible \*-ring is weak \*-reversible. The converse is true while the \*-ring has semi-proper involution \* and central \*-reflexive, we have the subsequent proposition.

**Proposition 11:** If  $R$  \*-ring has semi-proper involution \* and weak \*-reversible, then  $R$  is \*-weak \*-reversible.

**Proof.** Let  $ab = 0 = ab^*$  for some  $a, b \in R$ . Then,  $(Rbra)R(Rbra)^* = Rbra(Ra^*r^*)b^*R \subseteq RbraRb^*R = 0$ , for all  $r \in R$ , from semi-proper involution \*. Hence,  $R$  is \*-weak \*-reversible.

Recall that, [(U. A. Aburawash & Saad, 2019) and (U. A. Aburawash & Abdulhafed, 2018a)], if  $ab = ab^* = 0$  implies  $bRa = 0$  (resp.,  $bRa$  is central) for all  $a, b \in R$  is called \*-reflexive (resp., central \*-reflexive) \*-ring. Every \*-reversible is \*-reflexive ring with involution.

**Proposition 12:** If  $R$  is weak \*-reversible and central \*-reflexive, then  $R$  is \*-weak \*-reversible since  $R$  is a \*-ring.

**Proof.** Let  $ab = 0 = ab^*$  for some  $a, b \in R$ . Then  $(Rbra)^2 = RbraRbra = RRbrabra = 0$ ,  $(Rbra)(Rbra)^* = Rbraa^*r^*b^*R = Ra^*r^*brab^*R = 0$ , for all  $r \in R$ , from central \*-reflexive. Hence,  $R$  is \*-weak \*-reversible.

Now, prove the next result it is easy.

**Proposition 13.** The \*-weak \*-reversible ring with involution class is closed under taking subrings with involution and under direct sums (with changeless involution).

Now, the next proposition is similar to **Proposition 4**.

**Proposition 14.**  $R$  is a commutative \*-ring,  $T_{n \times n}(R)$  is \*-weak \*-reversible, with involution \* from **Proposition 4**.

In example 6 of (**U. A. Aburawash & Abdulhafed, 2018b**) there exists a \*-weak \*-reversible and quasi \*-IFP \*-ring which is not \*-reversible.

Next, for example, we note that the condition  $T_{n \times n}(R)$  of Proposition 14, cannot be weakened to the whole matrix \*-ring  $M_{n \times n}(R)$ , since  $n$  is more than 1.

**Example 3.** If  $R$  is a  $*$ -weak  $*$ -reversible  $*$ -ring and  $n$  is more than 1, then  $M_{2\times 2}(R)$ , with adjoint involution, is not  $*$ -weak  $*$ -reversible. For  $X = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$  and  $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,

we have  $XY = 0 = XY^*$  and for  $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}(R)$ , we see that  $RYZX = \begin{pmatrix} 0 & 2a \\ 0 & 2c \end{pmatrix}$  is not  $*$ -nil,  $\forall a, c \in R$ .

The subsequent result shows that central  $*$ -reversible  $*$ -rings lie properly between  $*$ -reversible and  $*$ -weak  $*$ -reversible rings with involution.

**Theorem 2.** Consider the following conditions, where  $R$  is a  $*$ -ring.

1.  $R$  is  $*$ -reversible.
2.  $R$  is central  $*$ -reversible.
3.  $R$  is  $*$ -weak  $*$ -reversible.
4.  $R$  is weak  $*$ -reversible.

**Proof. 1  $\Rightarrow$  2:** It's clear.

**2  $\Rightarrow$  3:** If  $ab = 0 = ab^*$ , then  $r_1ab = 0 = r_1ab^*$  for all  $a, b, r_1 \in R$ , and  $br_1a$  is central, since  $R$  is central  $*$ -reversible. Hence,  $(rbr_1a)^2 = rbr_1arbr_1a = rrbr_1abr_1a = 0$  and  $(rbr_1a)(rbr_1a)^* = rbr_1aa^*r_1^*b^*r^* = ra^*r_1^*br_1ab^*r^* = 0$  for all  $r \in R$  and  $R$  is weak  $*$ -reversible.

**3  $\Rightarrow$  4:** It's clear.

By [(U. A. Aburawash & Abdulhafed, 2018b), Propositions 3, 4] and Propositions (11 and 12) the converse of Theorem 2 isn't true.

Further, we get corollaries from [(U. A. Aburawash & Abdulhafed, 2018b), Proposition 4] and Theorem 2.

**Corollary 5.** Every  $*$ -domain is a  $*$ -weak  $*$ -reversible  $*$ -ring.

**Corollary 6.** If  $R$  is a central  $*$ -reversible and  $*$ -Baer ring with involution, then  $R$  is  $*$ -weak  $*$ -reversible.

It's important to remember that every Bear  $*$ -ring is a  $*$ -Bear  $*$ -ring. We have the following result.

**Corollary 7.** If  $R$  is a Baer and central  $*$ -reversible  $*$ -ring, then  $R$  is  $*$ -weak  $*$ -reversible.

Now, as shown by the next theorem, the central  $*$ -reversible  $*$ -rings properly lie between the classes of  $*$ -reversible, weak quasi- $*$ -IFP, and  $*$ -weak quasi- $*$ -IFP\* $*$ -rings

**Theorem 3.** Consider the following conditions, where  $R$  is a  $*$ -ring.

1.  $R$  is  $*$ -reversible.
2.  $R$  is central  $*$ -reversible.
3.  $R$  is weak quasi- $*$ -IFP.
4.  $R$  is  $*$ -weak quasi- $*$ -IFP.

**Proof. 1  $\Rightarrow$  2:** Obviously.

**2  $\Rightarrow$  3:** By [(U. A. Aburawash & Abdulhafed, 2018b), Theorem 2].

**3  $\Rightarrow$  4:** From Propositions (5 and 6).

The converse of Theorem 3 is not true from [(U. A. Aburawash & Abdulhafed, 2018b), Example 6].

The following results, from [(U. A. Aburawash & Saad, 2019), **Proposition 10**], [(U. A. Aburawash & Abdulhafed, 2018b), **Corollaries (7, 8, 9, 10)**], and **Theorem 2**.

**Corollary 8.** If  $\mathbf{R}$  is  $*$ -ring and central reduced then  $\mathbf{T}(\mathbf{R}; \mathbf{R})$  is a  $*$ -weak  $*$ -reversible.

**Corollary 9.** If the  $*$ -ring  $\mathbf{R}$  is reduced then  $\mathbf{T}(\mathbf{R}; \mathbf{R})$  is a  $*$ -weak  $*$ -reversible.

**Corollary 10.** If the  $*$ -ring  $\mathbf{R}$  is  $*$ -reduced and  $*$ -reversible, then  $\mathbf{T}(\mathbf{R}; \mathbf{R})$ , with componentwise involution, is  $*$ -weak  $*$ -reversible.

**Corollary 11.** If the  $*$ -ring  $\mathbf{R}$  is reduced and  $*$ -reversible, then  $\mathbf{T}(\mathbf{R}; \mathbf{R})$ , with component wise involution, is  $*$ -weak  $*$ -reversible.

Thus, we note that by the diagram, of  $*$ -rings.

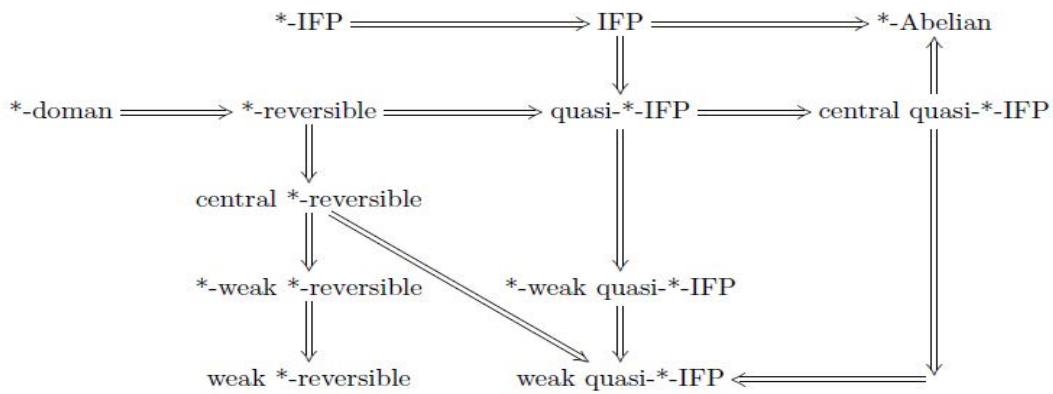


Diagram (3)

### Extending of $*$ -weak ( $*$ -IFP, quasi $*$ -IFP, and $*$ -reversible) rings with involution.

We will here focus on the properties of  $*$ -weak ( $*$ -IFP, the quasi  $*$ -IFP with involution, and  $*$ -reversible) rings with involution is proven to be extended from a  $*$ -ring to its localization and Laurent polynomial, the Dorroh extension, and from Ore  $*$ -ring to its classical quotient.

If  $\Omega^{-1}\mathbf{R} = \{u^{-1}a \mid u \in \Omega, a \in \mathbf{R}\}$ , then  $\Omega^{-1}\mathbf{R}$  is a  $*$ -ring with involution  $\diamond$  defined as  $(u^{-1}a)^\diamond = u^{-1}a^* = u^{-1}a^*$  where  $\mathbf{R}$  is a ring with involution,  $\Omega$  is a multiplicative in  $\mathbf{R}$  consisting of  $\mathbf{R}$  and central regular elements, see [(U. A. Aburawash & Abdulhafed, 2018b)].

For it, we have the propositions.

**Proposition 15.** If  $e$  is a central projection of  $\mathbf{R}$  since  $\mathbf{R}$  is a  $*$ -ring, then it's equivalent.

1.  $\mathbf{R}$  is  $*$ -weak quasi-\*IFP.
2.  $e\mathbf{R}$  and  $(1-e)\mathbf{R}$  are  $*$ -weak quasi-\*IFP.
3.  $\Omega^{-1}\mathbf{R}$  is  $*$ -weak quasi-\*IFP.

**Proof.**  $1 \Leftrightarrow 2$ : It is clear that this is straightforward since subrings with involution and finite direct products of  $*$ -weak quasi-\*IFP ring with involution is  $*$ -weak quasi-\*IFP.

$3 \Rightarrow 1$ : It is clear since  $\mathbf{R}$  is a subring with involution of  $\Omega^{-1}\mathbf{R}$ .

**1  $\Rightarrow$  3:** Since  $\mathbf{R}$  is a  $*$ -subring of  $\Omega^{-1}\mathbf{R}$ . Let  $\alpha\beta = 0 = \alpha\beta^*$  with  $\alpha = u^{-1}a, \beta = v^{-1}b, u, v \in \Omega$  and  $a, b \in \mathbf{R}$ , and let  $\gamma = w^{-1}c$  for any element of  $\Omega^{-1}\mathbf{R}, w \in \Omega, c \in \mathbf{R}$ . Since  $\Omega$  contained in the center of  $\mathbf{R}$ , we have  $0 = \alpha\beta = u^{-1}\alpha v^{-1}b = (u^{-1}v^{-1})ab$ , and  $0 = \alpha\beta^* = u^{-1}a(v^{-1}b)^* = (u^{-1}(v^{-1})^*)ab^*$ , and hence  $ab = 0 = ab^*$ , but  $\mathbf{R}$   $*$ -weak quasi- $*$ -IFP, like this for some  $n$  and  $m$  in positive integers that  $(acb)^n = 0, ((acb)(acb)^*)^m = 0$ . Thus  $(\alpha\gamma\beta)^n = (u^{-1}aw^{-1}cv^{-1}b)^n = ((vwa)^{-1}acb)^n = ((vwa)^{-1}n(acb)^n = 0)$  and  $((\alpha\gamma\beta)(\alpha\gamma\beta)^*)^n = ((u^{-1}aw^{-1}cv^{-1}b)(\beta^*\gamma^*\alpha^*))^n = ((u^{-1}aw^{-1}cv^{-1}b)(\beta^*\gamma^*\alpha^*))^m = ((vwa)^{-1}(acb)(b^*(v^{-1})^*c^*(w^{-1})^*a^*(u^{-1})^{-1})^m = ((vwa)^{-1}(acb)(u^*w^*v^*)^{-1}(b^*c^*a^*))^m = ((vwa)^{-1}(u^*w^*v^*)^{-1}(acb)(b^*c^*a^*))^m = ((vwa)^{-1}(u^*w^*v^*)^{-1})^m((acb)(acb)^*)^m$ . Therefore,  $\Omega^{-1}\mathbf{R}$  is  $*$ -weak quasi- $*$ -IFP.

**Proposition 16.** If  $\mathbf{R}$  is a  $*$ -ring, and  $e$  has a central projection in  $\mathbf{R}$ , the following is its equivalent:

1.  $\mathbf{R}$  is  $*$ -weak  $*$ -reversible.
2.  $e\mathbf{R}$  and  $(1 - e)\mathbf{R}$  are  $*$ -weak  $*$ -reversible.
3.  $\Omega^{-1}\mathbf{R}$  is  $*$ -weak  $*$ -reversible.

**Proof.** It is similar to that of **Proposition 15**.

We get results from [(U. A. Aburawash & Abdulhafed, 2018b), Proposition 6].

**Corollary 12.** Let  $\mathbf{R}$  be a  $*$ -ring, and  $e$  is a central projection of  $\mathbf{R}$ . Then  $e\mathbf{R}$  and  $(1 - e)\mathbf{R}$  are  $*$ -reversible if  $\mathbf{R}$  is  $*$ -weak  $*$ -reversible.

**Corollary 13.** If  $\mathbf{R}$  is  $*$ -reversible a  $*$ -ring and  $e$  central of  $e^2 = e = e^*$  in  $\mathbf{R}$ . Then  $e\mathbf{R}$  and  $(1 - e)\mathbf{R}$  are  $*$ -weak  $*$ -reversible.

Then, the ring with involution of the Laurent polynomials in  $x$ , with coefficients in the  $*$ -ring  $\mathbf{R}$ , consists of all formal sums  $f(x) = \sum_{i=k}^n a_i x^i$  an involution  $*$ ,  $f^*(x) = \sum_{i=k}^n a_i^* x^i$  with explicit addition and multiplication, where  $a_i \in \mathbf{R}$  and  $k, n$  (possibly negative) integers. We denote this plate by  $\mathbf{R}[x; x^{-1}]$ .

**Corollary 14.** For a  $*$ -ring  $\mathbf{R}$ ,  $\Omega$  is  $*$ -weak quasi- $*$ -IFP iff  $\mathbf{R}[x; x^{-1}]$  is  $*$ -weak quasi-IFP with involution.

**Proof.** It is sufficient to establish the necessity since  $\mathbf{R}[x]$  is a subring with involution of  $\mathbf{R}[x; x^{-1}]$ . Let  $\Omega = \{1, x, x^2, x^3, x^4, \dots\}$ . Then  $\Omega$  is a multiplicative closed subset with the involution of  $\Omega$ . Where  $\mathbf{R}[x; x^{-1}] = \Omega^{-1}\mathbf{R}[x]$ , it follows that  $\mathbf{R}[x; x^{-1}]$  is  $*$ -weak quasi- $*$ -IFP by Proposition 15.

**Corollary 15.** For  $\mathbf{R}$   $*$ -ring,  $\Omega$  is  $*$ -weak  $*$ -reversible if and only  $\mathbf{R}[x; x^{-1}]$  is  $*$ -weak  $*$ -reversible.

**Proof.** Like **Corollary 14** using Proposition 16.

Accordingly, we have the equivalence on  $*$ -weak quasi- $*$ -IFP and  $*$ -weak  $*$ -reversibility of another situation.

**Corollary 16.** If  $\mathbf{R}$  is a  $*$ -ring the following is its equivalent:

- 1)  $\mathbf{R}$  is  $*$ -weak quasi-IFP with involution.
- 2)  $\mathbf{R}[x]$  is  $*$ -weak quasi-IFP with involution.
- 3)  $\mathbf{R}[x; x^{-1}]$  is  $*$ -weak quasi-IFP with involution.

**Corollary 17.** For a  $*$ -ring  $R$ , the following is its equivalent:

1.  $R$  is  $*$ -weak  $*$ -reversible.
2.  $R[x]$  is  $*$ -weak  $*$ -reversible.
3.  $R[x; x^{-1}]$  is  $*$ -weak  $*$ -reversible.

Then, for the Dorroh expansion of the  $*$ -ring  $D(R, \mathbb{Z}) = \{(r, n) : r \in R, n \in \mathbb{Z}\}$   $R$  is a ring whose functions are  $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$  and  $(r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2)$ . The  $R$  involution can naturally extend to  $D(R, \mathbb{Z})$  in the form  $(r, n)^* = (r^*, n)$  (see (U. A. Aburawash, 1997)).

Next, we get the following results by [(U. A. Aburawash & Abdulhafed, 2018b), **Proposition 21**]:

**Corollary 18.** A  $*$ -ring  $R$  is  $*$ -reversible, and the Dorroh extension  $D(R, \mathbb{Z})$  of  $R$  is  $*$ -weak- $*$ -reversible.

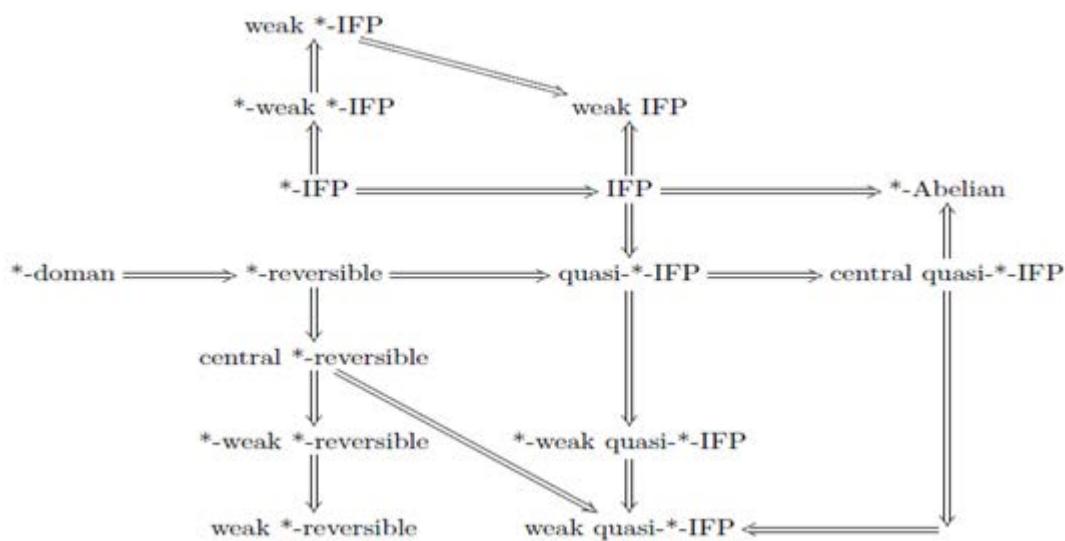
**Corollary 19.** A  $*$ -ring  $R$  is Dorroh extension  $D(R, \mathbb{Z})$  is  $*$ -reversible,  $R$  is  $*$ -weak- $*$ -reversible.

We know that Let  $R$  be a ring with involution  $*$  which is a left order in a ring  $Q$ . Then  $R$  is a right order in  $Q$  and  $Q$  has an involution given by  $(a^{-1}b)^* = b^*(a^*)^{-1}$  (see [(Martindale & 3rd, 1969), Lemma 4]), and [(U. A. Aburawash & Abdulhafed, 2018b), Theorem 4] given the following results.

**Corollary 20.** If  $R$  is  $*$ -reversible, then  $Q$  is  $*$ -weak- $*$ -reversible.

**Corollary 21.** If  $Q$  is  $*$ -reversible, then  $R$  is  $*$ -weak- $*$ -reversible.

## CONCLUSION



Here, we conclude the paper's results using diagrams to explain the relations among the corresponding classes, from diagrams [1], [2] and diagram [3].

Furthermore, we have the following conclusions:

Every  $*$ -IFP (resp., quasi  $*$ -IFP and  $*$ -reversible) are  $*$ -weak  $*$ -IFP (resp.,  $*$ -weak quasi  $*$ -IFP and  $*$ -weak- $*$ -reversible) rings with involution, also, each  $*$ -IFP is  $*$ -weak quasi  $*$ -IFP ring with involution. Moreover, every  $*$ -Doman is  $*$ -weak  $*$ -reversible ring with involution.

Finally, we future studies about properties of  $*$ -weak  $*$ - rings of ( $*$ -reflexive, S.  $*$ -reversible, and S.  $*$ -reflexive) rings with involution.

**Duality of interest:** The authors declare that they have a duality of interest associated with this manuscript.

**Author contributions:** The first author (Muna E. Abdulhafed) developed the theoretical formalism, and performed the analytic calculations into the final version of the manuscript. The second author (Aafaf E. Abduelhafid) collected the data, performed the analytic calculations, and analyzed the data.

**Funding:** The authors declare that we have never received any funding for the work reported in our manuscript.

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