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A multi-step formable transform decomposition method for solving fractional order Riccati equation

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Abstract

A multi-step formable transform decomposition method (MFTDM) is suggested in this study to solve the nonlinear fractional-order Riccati problem. It is well understood that a corresponding numerical solution given by the FTDM is only valid for a short period.In the case of integer-order systems, however, the MFTDM solutions are more correct and reliable throughout time and are in very good agreement with the exact solutions. The fractional derivative is described in the Caputo sense. The method is tested on prominent examples, and the results show that it is accurate and efficient when compared to other numerical methods.

Keywords: Caputo derivative; Fractional orderRiccati equation; Multi-step Formable transform decomposition Method.

INTRODUCTION

The formable transform decomposition method (FTDM) is a computational and analytical approach for solving fractional partial differential problems (Saadeh et al., 2023). The method provides the solution in terms of convergent series with easily computable components. In the past years, several academics have concentrated their efforts on the numerical solution of ordinary differential equations of fractional order and various numerical techniques, including the Fourier transform method (Kemle& Beyer, 1997), Homotopy perturbation method (Wang, 2007; Odibat & Momani, 2008; Mtawal & Alkaleeli, 2020), Homotopy analysis method (Canget al., 2009; Zurigatet al., 2010; Freihat, et al., 2014), Residual power series method (Ali et al., 2017), Alternative variational iteration method (Mtawalet al., 2020), Triple Shehu transform method (Alkaleeliet al., 2021; Kapooret al., 2022), the Laplace residual power series method (Burqan, et al., 2022),. Recently, a formable transformation decomposition method (FTDM) was applied by (Al-ZouBi&Zurigat, 2014), it combines the formable integral transform (Saadeh, 2021) and the decomposition method (Momani& Al-Khaled, 2005; Shawagfeh, 2002; Khanet al, 2013; Mahdyet al., 2015). In the present study, we analyze the suitability and value of the MFTDM as a method to obtain the right approximation solutions to the fractional differential equation of the form using a series of intervals.

$$D_*^{\beta} y(t) = g(t) - L[y(t)] - N[y(t)], \qquad (1)$$

with $t \geq 0,\ 0 < \alpha \leq 1,$ and subject to the initial condition.

$$y(0) = c. \qquad (2)$$

Where $D_*^{\beta} y(t)$ is the fractional derivative of Caputo?This optimized approach is known as the multi-step formable transform decomposition method.The MFTDM was successfully shown to effectively, quickly, and accurately solve fractional differential equations. There is an enormous class of nonlinear fractional differential equations with approximations that rapidly converge to exact solutions. We provided two examples to show the effectiveness of our results.

PRELIMINARIES

This section describes the essential terminology and notations used in the fractional derivative field (Caputo, 1969; Miller &Roos, 1993; Beyer &Konuralp, 1995; Gorenflo&Mainardi, 1997; Podlubny, 1999). Also covered are the definition and characteristics of the formable integral transform (Kanwal*et al*., 2018; Saadeh& Ghazal, 2021; Saadeh*et al*., 2023).

Definition 2.1. The Riemann-Liouville fractional integral of order $\beta > 0$,ofa function $f \in C_{\rho}, \rho \geq -1$ is defined as (Miller &Roos, 1993; Beyer &Konuralp, 1995; Gorenflo&Mainardi, 1997; Podlubny, 1999) :

$$J^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} f(\zeta)\, d\zeta .$$

Definition 2.2. Let $f \in C_n^m, m \in N \cup \{0\}$. The Caputo fractional derivative of f in the Caputo sense is defined as follows (Caputo, 1969):

$$D_t^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\zeta)^{m-\beta-1} f^{(m)}(\zeta)\ d(\zeta), \\ \qquad\qquad\qquad m-1 < \beta \leq m, \\ D_t^{\beta} f(t), \qquad\qquad \beta = m. \end{cases}$$

Definition 2.3.(Saadeh& Ghazal, 2021; Saadeh*et al*., 2023) A function $f : [0, \infty) \to R$ is said to be of exponential order $\beta\,(\beta > 0)$, if there $|f(t)| \leq M\ e^{\beta t}$, for all $t \geq t_0$.

Definition 2.4. The formable integral transform of a continuous function f on the interval $(0,\infty)$ is defined by (Saadeh& Ghazal, 2021; Saadeh*et al*., 2023)

$$F[f(t)] = \frac{s}{u} \int_0^{\infty} \exp\left(-\frac{st}{u}\right) f(t)\, dt$$
$$= A(s,u).$$

The formula for the inverse formable integral transform is

$$f(t) = F^{-1}[A(s,u)]$$
$$= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{s} \exp\left(\frac{st}{u}\right) A(s,u)\, ds.$$

A constant's or polynomial's formable integral transform is given by:

$$F[a] = a.$$

$$F\left[t^{m}\right]=\left(\frac{u}{s}\right)^{m} m!, m \in N .$$

$$F\left[t^{\beta}\right]=\left(\frac{u}{s}\right)^{\beta} \Gamma(\beta+1), \beta>0.$$

Theorem 2.1. The Mittag-Leffler function's formable integral transform is provided by(Kanwal*et al.*, 2018)

$$A(s,u)=\sum_{i=0}^{\infty} \lambda^{i}\left(\frac{u}{s}\right)^{\beta i+\alpha-1} .$$

Theorem 2.2. Let f be a piecewise continuous function defined on $[0, \infty)$. Then, the formable integral transform of the Riemann-Liouville fractional integral of order $\beta>0$ of the function f is given by (Saadeh& Ghazal, 2021; Saadeh*et al.*, 2023)

$$F\left[I_{t}^{\beta} f(t)\right]=\left(\frac{u}{s}\right)^{\beta} A(s,u).$$

Theorem 2.3. Let f be a piecewise continuous function defined on $[0, \infty)$. Then, the formable integral transform of the Caputo fractional derivative of the order $\beta, m-1<\beta \leq m$, of the function f is given by (Saadeh& Ghazal, 2021; Saadeh*et al.*, 2023)

$$F\left[D_{t}^{\beta} f(t)\right]=\left(\frac{u}{s}\right)^{\beta}[A(s,u)] -\left(\frac{u}{s}\right)^{\beta}\left(\sum_{i=0}^{m-1}\left(\frac{u}{s}\right)^{i} f^{(i)}(0)\right).$$

MATERIALS AND METHODS

Despite the fact that the FTDM (Saadeh, 2021) is used to approximate solutions to a wide range of nonlinear problems in terms of convergent series with readily calculated components, it has certain shortcomings: The series solution always converges rapidly in a small region and slowly in a bigger region. In this section, we present the core notions of the MFDTM that we built for numerically solving our problems (1) and (2). It is a simple tweak to regular FTDM that verifies the accuracy of the estimated solution for large time intervals. The solution is expanded over the interval $[0, t]$by dividing it into$i$ - subintervals $[t_{i-1}, t_{i}],\ j=1,2, \ldots, i$ of equal length Δt . If$t^{*}$is the initial value and$y_{j}(t)$ is an approximation in each subinterval $[t_{i-1}, t_{i}],\ j=1,2, \ldots, i$, the equations (1) and (2) can be transformed into the following system:

$$D_{*}^{\beta} y_{j}(t)=g(t)-L\left[y_{j}(t)\right]-N\left[y_{j}(t)\right], \qquad (3)$$

with $t \geq 0,\ 0<\beta \leq 1,\quad j=1,2, \ldots, i$ and subject to the initial condition

$$y_{j}(t^{*})=a, \quad y_{j}(t^{*})=y_{j-1}(t_{j-1})=c_{j}, \qquad (4)$$

where$D_{*}^{\beta} y_{j}(t)$ is the Caputo fractional derivative,the source term is $g(t)$, L means the linear differential operator and N means the general nonlinear differential operator.
Using the formable integral (denoted by F in this study) on both sides of Equation (3), we obtain

$$F\left[D_*^{\beta} y_j(t)\right]=F\left[g(t)-L\left[y_j(t)\right]\right.$$
$$\left.-N\left[y_j(t)\right]\right]. \qquad (5)$$

Equation (5) can be read using Theorem 2.3 and the initial condition in equation (4) as

$$F\left[y_j(t)\right]=c_j+\left(\frac{u}{s}\right)^{\beta} F\left[g(t)\right.$$
$$\left.-L\left[y_j(t)\right]-N\left[y_j(t)\right]\right]. \qquad (6)$$

Using the formable inverse on both sides of Equation (6) yields

$$y_j(t)=F^{-1}\left[c_j\right]$$
$$+F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[g(t)\right]-L\left[y_j(t)\right]\right] \qquad (7)$$
$$-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[N\left[y_j(t)\right]\right]\right].$$

Then,

$$y_j(t)=\sum_{i=0}^{n} y_{j,i}(t), \qquad (8)$$

The nonlinear term in Equation (7) can be decomposed as follows:

$$N\left[y_{j,i}(t)\right]=\sum_{i=0}^{\infty} A_{j,i}(y_{j,0},y_{j,1},\ldots,y_{j,i}), \qquad (9)$$

for some Adomian's polynomials A_i that are given by (Ghorbani, 2009).

$$A_{j,i}(y_{j,0},y_{j,1},\ldots,y_{j,i})=$$
$$\frac{1}{i!}\left(\sum_{i=0}^{n}\frac{d^{i}}{d\lambda^{i}}N\left(\sum_{k=0}^{\infty}\lambda^{i}\, y_{j,i}\right)\right)_{\lambda=0}. \qquad (10)$$

Substituting Equations (8) and (9) into Equation (7) yields

$$\sum_{i=0}^{n} y_{j,i}(t)=c_j+F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[g(t)\right]\right]$$
$$-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[\sum_{i=0}^{\infty}\left(L\left[y_{j,i-1}(t)\right]\right)\right]\right] \qquad (11)$$
$$-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[\sum_{i=0}^{\infty}\left(A_{j,i-1}\right)\right]\right].$$

After considering the comparison in Equation (11), we obtain

$$y_{j,0}(t)=c_j,$$

$$y_{j,1}(t)=F^{-1}\left[\left(\frac{u}{s}\right)^{\beta}F\left[g(t)\right]\right]$$

$$-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta}F\left[L\left[y_{j,0}(t)\right]\right]\right]$$

$$-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta}f\left[A_{j,0}\right]\right],$$

$$y_{j,2}(t)=-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta}F\left[L\left[y_{j,1}(t)\right]\right]\right]$$

$$-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta}F\left[A_{j,1}\right]\right],$$

$$\vdots$$

$$y_{j,i}(t)=-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta}F\left[L\left[y_{j,i-1}(t)\right]\right]\right]$$

$$-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta}F\left[A_{j,i-1}\right]\right]. \qquad (12)$$

In addition, a power series solution needs the form

$$y_j(t)=\sum_{i=0}^{\infty} y_{j,i}(t), \quad j=1,\ 2,\ldots,i. \,(13)$$

Finally, the system (1) solutions have the form

$$y(t)=\begin{cases} y_1(t), & t\in\left[t_0,t_1\right], \\ y_2(t), & t\in\left[t_1,t_2\right], \\ \vdots & \\ y_i(t), & t\in\left[t_{i-1},t_i\right]. \end{cases} \qquad (14)$$

RESULTS

To show the applicability and effectiveness of our approach for solving nonlinear fractional Riccate equations, we explore the following examples: (Al-ZouBi&Zurigat, 2014;Zurigatet al., 2010; Canget al.,2009) :

Example 1.Consider the following nonlinear fractional Riccati equation

$$D_*^{\beta}y(t)=1-y^2(t), \quad t\geq 0, \quad 0<\beta\leq 1, \qquad (15)$$

with the initial condition

$$y(0)=0. \qquad (16)$$

The exact solutions of this equation when $\beta=1$ is $y(t)=\frac{e^{2t}-1}{e^{2t}+1}$. If $y_j(t)$ is an approximation in each subinterval $\left[t_{i-1},t_i\right]$, $j=1,2,\ldots,i$, the equations (15) and (16) can be transformed into the following system:

$$D_*^{\beta} y_j(t) = 1 - y_j^{\,2}(t), \quad j = 1, 2, \ldots, i. \quad (17) \qquad y_j(t^*) = c_j, \quad (18)$$

With initial conditionwith $c_1 = 1.$

Using the formable integral (denoted by F in this study) on both sides of Equation (17), we obtain

$$F\left[D_*^{\beta} y_j(t)\right] = F\left[1 - y_j^{\,2}(t)\right], \quad (19)$$

Equation (19) can be read using Theorem 2.3 and the initial condition in equation (18) as

$$F\left[y_j(t)\right] = c_j + \left(\frac{u}{s}\right)^{\beta} F\left[1 - y_j^{\,2}(t)\right]. \quad (20)$$

Using the formable inverse on both sides of Equation (20) yields

$$y_j(t) = F^{-1}\left[c_j\right] + F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[1 - y_j^{\,2}(t)\right]\right]. \quad (21)$$

Where $N\left[y_j\ (t)\right] = y_j^2(t)$ is a nonlinear operator, respectively.The nonlinear term of Eq.(21) can be decomposed as

$$N\left[y_j\ (t)\right] = y_j^2(t) = \sum_{i=0}^{\infty} A_{j,i}. \quad (22)$$

Adomian polynomials' first few components are provided by

$$A_{j,0} = y_{j,0}^2,$$
$$A_{j,1} = 2 y_{j,0}\, y_{j,1},$$
$$A_{j,2} = 2 y_{j,0}\, y_{j,2} + y_{j,1}^2.$$
$$\vdots$$

Assume that the solution of Equation (17) has the following series

$$y_j\ (t) = \sum_{i=0}^{n}\ y_{j,i}(t). \quad (23)$$

Substituting Equations (22) and (23) into Equation (21) yields

$$\sum_{i=0}^{n} y_{j,i}(t) = c_j + F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[1 - \sum_{i=0}^{\infty} A_{j,i}\right]\right]. \quad (24)$$

After considering the comparison in Equation (24), we obtain

$$y_{j,0}(t) = c_j,$$
$$y_{j,1}(t) = F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[1 - A_{j,0}\right]\right]$$
$$= \frac{\left(1 - c_j^2\right)(t - t^*)^{\beta}}{\Gamma(\beta + 1)},$$
$$y_{j,2}(t) = -F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[A_{j,1}\right]\right]$$
$$= \frac{-2c_j\left(1 - c_j^2\right)(t - t^*)^{2\beta}}{\Gamma(2\beta + 1)},$$

$$y_{j,3}(t) = -F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[A_{j,2}\right]\right]$$
$$= \frac{4c_j^2\left(1-c_j^2\right)(t-t^*)^{3\beta}}{\Gamma(3\beta+1)}$$
$$-\frac{\left(1-c_j^2\right)^2 \Gamma(2\alpha+1)(t-t^*)^{3\beta}}{\Gamma^2(\beta+1)\Gamma(3\beta+1)}.$$
$$\vdots \qquad (25)$$

The series solution to equation (17) is provided by

$$y_j(t) = c_j + \frac{\left(1-c_j^2\right)(t-t^*)^{\beta}}{\Gamma(\beta+1)}$$
$$-\frac{2c_j\left(1-c_j^2\right)(t-t^*)^{2\beta}}{\Gamma(2\beta+1)} + \frac{4c_j^2\left(1-c_j^2\right)(t-t^*)^{3\beta}}{\Gamma(3\beta+1)}$$
$$-\frac{\left(1-c_j^2\right)^2 \Gamma(2\beta+1)(t-t^*)^{3\beta}}{\Gamma^2(\beta+1)\Gamma(3\beta+1)} + \ldots.$$

In this example, the suggested method is applied to the interval $\left[0,10\right]$. We selected dividing the interval $\left[0,10\right]$ into subintervals with a time step of $\Delta t = 0.5$.Figures 1 and 2 show the series solution of the MFTDM of the nonlinear fractional Riccati equations (15) and (16) for $\beta = 1, 0.7, 0.9$ and the exact. The graphical results show that the results produced using the MFTDM for $\beta = 1$ very closely correspond to the results of the exact solution.This emphasizes MFTDM applicability to many different kinds of nonlinear fractional differential equations, as well as its reliability and promise when compared to existing methods. Furthermore, as in the preceding instance, the numerical results produced by the MFTDM have the same course for various values of β. All results are obtained using Maple 16.

Example 2. Consider the following nonlinear fractional Riccati equation

$$D_*^{\beta} y(t) = 1 + 2y(t) - y^2(t), \quad t \geq 0, \quad 0 < \beta \leq 1, (26)$$

with the initial condition

$$y(0) = 0. \qquad (27)$$

The exact solutions of this equation when $\beta = 1$ is $y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$. If $y_j(t)$ is an approximation in each subinterval $\left[t_{i-1}, t_i\right]$, $j = 1, 2, \ldots, i$, the equations (26) and (27) can be transformed into the following system:

$$D_*^{\beta} y_j(t) = 1 + 2y_j(t) - y_j^2(t), \quad j = 1, 2, \ldots, i. \quad (28)$$
$$y_j(t^*) = c_j, (29)$$

With initial conditionswith$c_1 = 1$.

Using the formable integral (denoted by F in this study) on both sides of Equation (28), we obtain

$$F\left[D_*^{\alpha} y_j(t)\right]=F\left[1+2y_j(t)-y_j^{\,2}(t)\right], (30)$$

Equation (30) can be read using Theorem 2.3 and the initial condition in equation (29) as

$$F\left[y_j(t)\right]=c_j+\left(\frac{u}{s}\right)^{\beta} F\left[1+2y_j(t)-y_j^{\,2}(t)\right]. (31)$$

Using the formable inverse on both sides of Equation (31) yields

$$\begin{aligned} y_j(t) &= F^{-1}\left[c_j\right] \\ &+F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[1+2y_j(t)\right]\right] \quad (32) \\ &-F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} f\left[y_j^{\,2}(t)\right]\right] \end{aligned}$$

Where $L\left[y_j(t)\right]=y_j(t), N\left[y_j(t)\right]=y_j^2(t)$ are linear and nonlinear operators, respectively.The nonlinear term of Eq.(32) can be decomposed as

$$N\left[y_j(t)\right]=y_j^2(t)=\sum_{i=0}^{\infty} A_{j,i}. \qquad (33)$$

Adomian polynomials' first few components are provided by

$$A_{j,0}=y_{j,0}^2,$$

$$A_{j,1}=2y_{j,0}\,y_{j,1},$$

$$A_{j,2}=2y_{j,0}\,y_{j,2}+y_{j,1}^2.$$

$$\vdots$$

Assume that the solution of Equation (28) has the following series

$$y_j(t)=\sum_{i=0}^{n} y_{j,i}(t). \qquad (34)$$

Substituting Equations (33) and (34) into Equation (32) yields

$$\sum_{i=0}^{n} y_{j,i}(t)=c_j+ \quad (35)$$

$$F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[1+2\,y_{j,i}-\sum_{i=0}^{\infty} A_{j,i}\right]\right].$$

After considering the comparison in Equation (35), we obtain

$$y_{j,0}(t)=c_j,$$

$$y_{j,1}(t)=F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[1+2\,y_{j,0}-A_{j,0}\right]\right]$$

$$=\frac{\left(1+2c_j-c_j^2\right)(t-t^*)^{\beta}}{\Gamma(\beta+1)},$$

$$y_{j,2}(t)=F^{-1}\left[\left(\frac{u}{s}\right)^{\beta} F\left[2\,y_{j,1}-A_{j,1}\right]\right]$$
$$=\frac{2\left(1-c_j\right)\left(1+2c_j-c_j^2\right)(t-t^*)^{2\beta}}{\Gamma(2\beta+1)} \qquad (36)$$
$$-\frac{\left(1+2c_j-c_j^2\right)^2\Gamma(2\beta+1)\,(t-t^*)^{2\beta}}{\Gamma^2(\beta+1)\,\Gamma(2\beta+1)},$$
$$\vdots$$

The series solution to equation (28) is provided by

$$y_j(t)=c_j+\frac{\left(1+2c_j-c_j^2\right)(t-t^*)^{\beta}}{\Gamma(\beta+1)}$$
$$+\frac{2\left(1-c_j\right)\left(1+2c_j-c_j^2\right)(t-t^*)^{2\beta}}{\Gamma(2\beta+1)}$$
$$-\frac{\left(1+2c_j-c_j^2\right)^2\Gamma(2\alpha+1)\,(t-t^*)^{2\beta}}{\Gamma^2(\beta+1)\,\Gamma(2\beta+1)}+\ldots.$$

In this example, the suggested method is applied to the interval $[0,10]$. We selected dividing the interval $[0,5]$ into subintervals with a time step of $\Delta t=0.1$.Figures 3 and 4 show the series solution of the MFTDM of the nonlinear fractional Riccati equations (26) and (27) for $\beta=1, 0.7, 0.9$ and the exact. The graphical results show that the results produced using the MFTDM for $\beta=1$ very closely correspond to the results of the exact solution.This emphasizes MFTDM applicability to many different kinds of nonlinear fractional differential equations, as well as its reliability and promise when compared to existing methods. Furthermore, as in the preceding instance, the numerical results produced by the MFTDM have the same course for various values of β. All results are obtained using Maple 16.

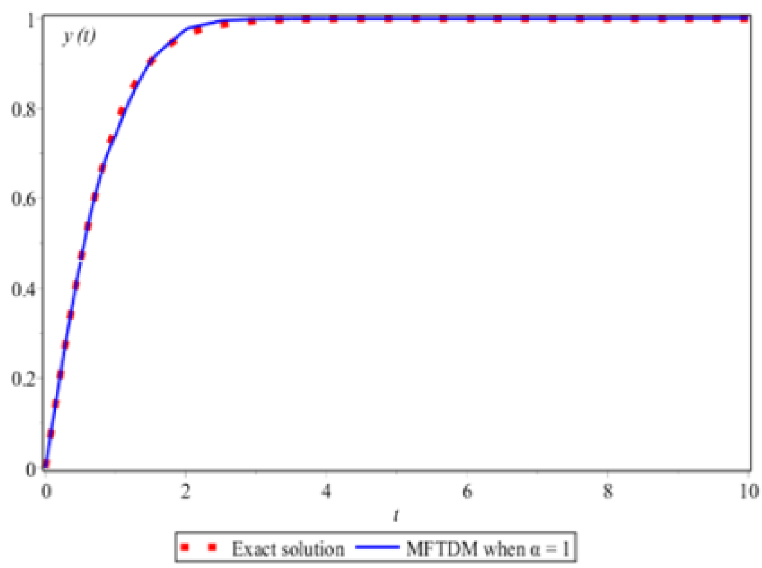

Figure: (1). Comparison between the exact and the MFTDM solutions of $y(t)$ for $\beta=1$.

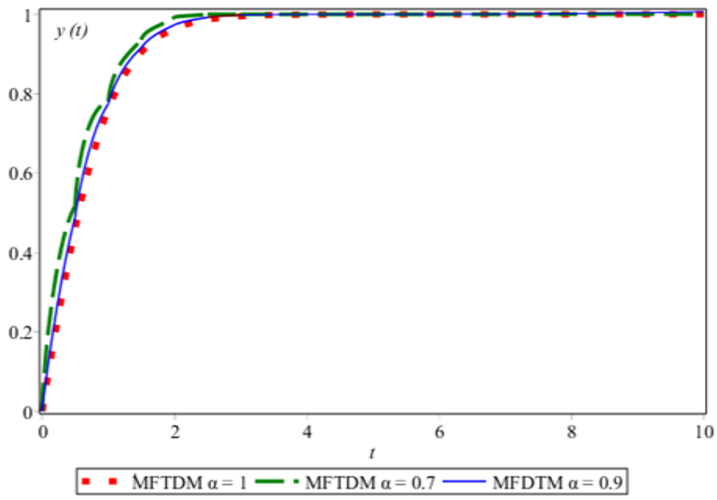

Figure: (2). The MFTDM solution of $y(t)$ for different values of β .

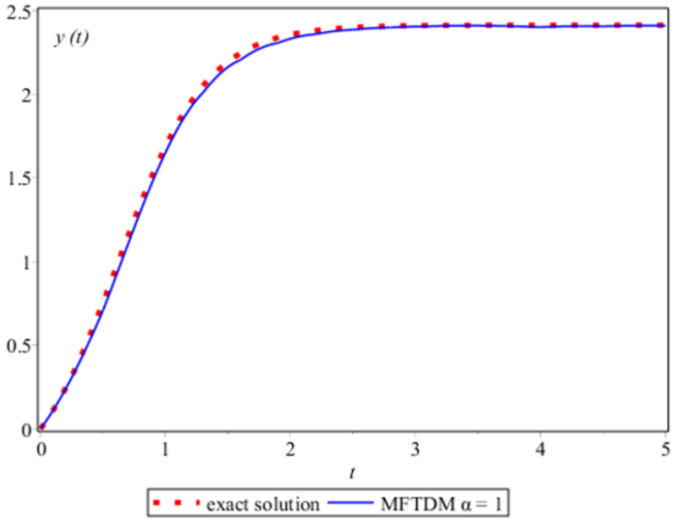

Figure: (3). Comparison between the exact and the MFTDM solutions of $y(t)$ for $\beta = 1$.

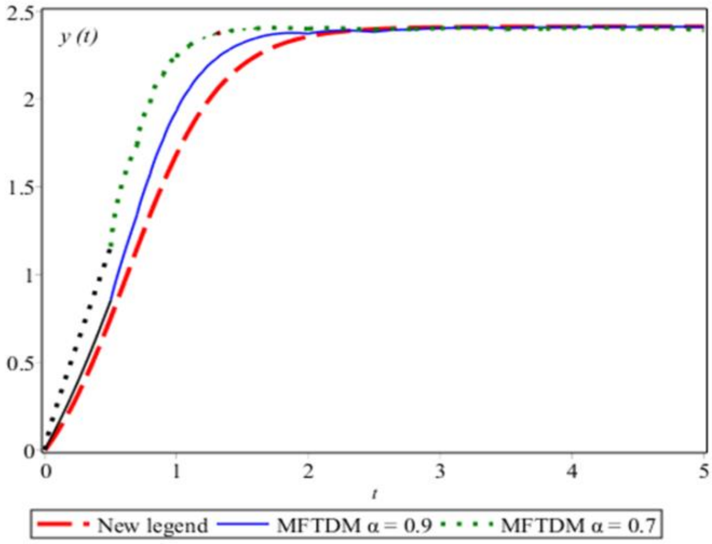

Figure: (4). The MFTDM solution of $y(t)$ for different values of β .

DISCUSSION

The graphical results show that the results produced using the MFTDM $\beta = 1$ very closely correspond to the results of the exact solution. This emphasizes the MFTDM applicability to many different kinds of nonlinear fractional differential equations, as well as its reliability and promise when compared to existing methods. Furthermore, as in the preceding instance, the numerical results produced by the MFTDM have the same course for various values of β. This is completely consistent with the research results of Al-Zoubi&Zurigat (2014).

CONCLUSION

A multi-step formable transform decomposition method (MFTDM) is suggested in this study to solve the nonlinear fractional-order Riccati problem.The MFTDM has been shown to solve fractional Riccati equations effectively, easily, and accurately. Approximate solutions quickly converge on exact solutions. This is completely consistent with the research results of Al-Zoubi&Zurigat (2014).Finally, we conclude that MFTDM is an excellent enhancement of existing numerical approaches.All results are obtained using Maple 16.

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