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The Conduct of Statistical Models when Parameters Are Perturbed From Their True Values

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Abstract

This paper aims to investigate the so-called non-linear properties of the skeleton of a non-linear autoregressive process, i.e., if $\underline{X}_t - f(\underline{X}_{t-1}) = \varepsilon_t$. Setting the variance of $\varepsilon_t = 0$, the skeleton of the process is obtained. Having fitted a self-exciting threshold autoregressive (SETAR) model to data and obtained a 95% confidence interval for parameters, we study the behaviour of non-linear properties, e.g., a limit cycle, amplitude dependency frequency, etc. We mainly consider a limit cycle for values of parameters at various locations within the confidence interval, or we may just slightly perturb model parameters from their true values.

Keywords: SETAR models, A limit point, A limit cycle, Simulation.

INTRODUCTION

A Self–Exciting Threshold Autoregressive (SETAR) Model: Tong (1983) proposed a self-exciting threshold autoregressive (SETAR) Model in his study of river flow data, which takes the form

$$X_t = \begin{cases} b^{(1)} X_{t-1} + \varepsilon_t^{(1)} & \text{if } X_{t-d} \le r \\ b^{(2)} X_{t-1} + \varepsilon_t^{(2)} & \text{if } X_{t-d} > r \end{cases} \qquad (1)$$

where $\varepsilon_t^{(1)}, \varepsilon_t^{(2)}$ are each a strict white-noise process, $b^{(1)}$, $b^{(2)}$ are constants, and d the delay parameters, respectively. The multiple threshold model takes the formula

$$X_t = b^{(j)} X_{t-1} + \varepsilon_t^{(j)} \text{ if } X_{t-d} \in R^{(j)}; j = 1,2,\dots,l$$

Where $R^{(1)}, R^{(2)},\dots, R^{(l)}$ are given subset of the real line R, which define a partition of R into disjoint intervals $(-\infty,r_0], [r_0,r_1],\dots, [r_{l-1},\infty)$, with $R^{(1)}$ denoting the interval $(-\infty,r_0]$ and $R^{(l)}$ denoting the interval $[r_{l-1},\infty)$ (Priestley, 1988).

Jones and Cox (1978) discussed the general first order non-linear model

$$X_t - f(X_{t-1}) = \varepsilon_t \qquad (2)$$

where f (.) is some general non-linear function, and ε_t is a sequence of independent variables. Model (1) can be used as a piecewise linear approximation to Model (2).

Similarly, a SETAR model of order $k > 1$ is given by

$$X_t = b_0^{(j)} + \sum_{i=1}^{k} b_i^{(j)} X_{t-i} + \varepsilon_t^{(j)} \text{ if } \underline{X}_{t-1} \in R^{(j)};\ j = 1,2,\ldots,l \qquad (3)$$

Where $R^{(j)}$ is a given region of k-dimensional Euclidean space $R^{(k)}$, and X_{t-1} $=(X_{t-1},X_{t-2},\ldots,X_{t-k})^T$ is the state vector at time $t-1$. Model (3) can be regarded as a piecewise linear approximation to the general k^{th} order non-linear AR model

$$\underline{X}_{t-1} - f(\underline{X}_{t-1}) = \varepsilon_t \qquad (4)$$

A Limit Cycle: According to Tong (1983), a limit cycle in discrete time can be defined as follows: Given a non-linear difference equation $X_t - f(X_{t-1}) = 0$, here $X_t = (X_{t-1},X_{t-2},\ldots,X_{t-p})$ is the state vector at time t, and f is a vector valued function, let $f^{(j)}$ denote the j^{th} iterate of f. Any vector of dimension p, which satisfies $f^{(jm)}(X) \rightarrow V$ as $j \rightarrow \infty$, is said to be a stable periodic point with a period m, with respect to domain $D \in R^p$. In this case $V_1, f^{(1)},\ldots, f^{(p-1)}(V_1)$, are all district stable limit points (Priestley, 1988).

The stable limit cycle is nothing but a set of vectors $(V_1,V_2,\ldots,V_{p-1})$. A limit cycle does depend only on the parameter of the system and the initial conditions of the system. A limit cycle is said to be stable if it does not change with changing the initial conditions of the system. A stable limit cycle is the only one that can be observed in practice being one of the models of behaviour to which the system, then, it is said to be robust. A limit cycle with an infinite period is known as chaos, which is very dependent on the initial values of the system. SETAR models can give rise to a limit cycle behaviour where the white-noise is suppressed, or equivalently, when it has zero variance (Tong, 1990).

MATERIALS AND METHODS

A Simulation Study: The aim of this study is to see whether, if we slightly change the parameters of a model that is known to have a limit cycle of a specific period, over their 95% bootstrap confidence intervals, the model would still have a limit cycle with a reasonable period, or would a period vary considerably as parameter values wander over their bootstrap confidence intervals. We consider simulation on two well-known models due to Tong and Lim (1980), which are known to have stable limit cycles each of period 9. The two models are

$$X_t = \begin{cases} 0.62 + 1.25X_{t-1} - 0.43X_{t-2} + \varepsilon_t^{(1)} & \text{if } X_{t-2} \leq 3.25 \\ 2.25 + 1.52X_{t-1} - 1.24X_{t-2} + \varepsilon_t^{(2)} & \text{if } X_{t-2} > 3.25 \end{cases} \quad (5)$$

$$X_t = \begin{cases} 0.546 + 1.032X_{t-1} - 0.173X_{t-2} + 0.171X_{t-3} - 0.431X_{t-4} \\ +0.332X_{t-5} - 0.284X_{t-6} + 0.210X_{t-7} + \varepsilon_t^{(1)} & \text{if } X_{t-2} \leq 3.16 \\ 2.632 + 1.492X_{t-1} - 1.32X_{t-2} + \varepsilon_t^{(2)} & \text{if } X_{t-2} > 3.16 \end{cases} \quad (6)$$

Our simulation consists of the following steps:

Step1. The bootstrap confidence interval of each parameter is constructed by using 50 replications, except for the delay parameter, where instead of constructing a bootstrap confidence interval, we allow the delay parameter to take the values 1, 2, 3, 4, and 5.

Step2. Starting from the lower limit of the bootstrap confidence interval of each parameter, and increasing by values, we simulate 1000 observations from each one of the skeletons of Models 5 and 6. The last 1000 observations are tested for a limit cycle with a possible maximum period of up to 500.

Step3. The experiment is terminated when the new perturbed value of each parameter reaches roughly the upper limit of its bootstrap confidence interval.

Step4. The above steps were repeated for each parameter in the skeleton of Models 5and 6.

We could, if we wished, instead of constructing the bootstrap confidence interval of each parameter in Step 1, just perturb the model parameters from their true value; therefore, Step 1 is not an essential step.

It should be mentioned that, in this study, we do not investigate each limit cycle in detail (i.e., how many sub-limit cycles it has or whether it is stable or not). We mainly investigate the existence of a limit cycle and its period. In cases where a no–limit cycle of a period less than or equal to 500 exists, we assume either it has a period greater than 500, or it is a chaos.

RESULTS

The value of each parameter, as it wanders over roughly its 95% bootstrap confidence interval, and the period of the corresponding limit cycle (if any) are given in Tables 1 to 7. Cases, where no limit cycle of a period less than, or equal to, 500 exists are denoted by $*$.

When the same limit cycle occurs in a sub-interval rather than at a single point within the bootstrap confidence interval of each parameter, the lower and upper limits of the sub-interval are given in each table. A single point is given as a sub-interval with equal upper and lower limits. Due to the similarity in the results for Models (5-6) and the lack of space, we give here only the results for Model 5. The other results can be obtained from the authors.

Table: (1). The limit cycles occurred due to small changes in the constant–term in the first region for the skeleton of Model 5.

Bootstrap Confidence Interval		Period
Lower Limit	Upper Limit	
0.474	0.584	1
0.585	0.585	12
0.586	0.588	11
0.589	0.592	10
0.593	0.600	9
0.601	0.610	8
0.611	0.638	9
0.639	0.739	8

Table: (2). The limit cycles occurred due to small changes in the first-coefficient in the first region for the skeleton of Model 5.

Bootstrap Confidence Interval		Period
Lower Limit	Upper Limit	
1.202	1.238	1
1.239	1.239	13
1.240	1.241	10
1.242	1.243	9
1.244	1.247	8
1.248	1.257	9
1.258	1.310	8
1.311	1.363	9

Table: (3). The limit cycles occurred due to small changes in the second-coefficient in the first region for the skeleton of Model 5.

Bootstrap Confidence Interval		Period
Lower Limit	Upper Limit	
-0.597	-0.464	1
-0.463	-0.463	25
-0.462	-0.462	12
-0.461	-0.461	77
-0.460	-0.450	*
-0.449	-0.417	9

Table:(4). The limit cycles occurred due to small changes in the constant-term in the second region for the skeleton of Model 5.

Bootstrap Confidence Interval		Period
Lower Limit	UpperLimit	
2.134	2.214	8
2.215	2.274	9
2.275	2.364	8
2.365	2.374	51
2.375	2.474	*
2.475	2.484	98
2.485	2.494	144
2.495	2.504	169
2.505	2.514	103
2.515	2.524	81
2.525	2.534	47
2.535	2.574	12

Table:(5). The limit cycles occurred due to small changes in the first-coefficient in the second region for the skeleton of Model 5.

Bootstrap Confidence Interval		Period
Lower Limit	Upper Limit	
1.340	1.410	10
1.411	1.484	9
1.485	1.510	8
1.511	1.530	9
1.531	1.553	8
1.554	1.554	34
1.555	1.555	51
1.556	1.556	17
1.557	1.557	70
1.558	1.566	*

Table: (6). The limit cycles occurred due to small changes in the second-coefficient in the second region for the skeleton of Model 5.

Bootstrap Confidence Interval		Period
Lower Limit	Upper Limit	
-1.367	-1.343	10
-1.342	-1.275	9
-1.274	-1.250	8
-1.249	-1.231	9
-1.230	-1.205	8
-1.204	-1.204	25
-1.203	-1.203	442
-1.202	-1.193	*

Table: (7). The limit cycles occurred due to small changes in the threshold value in for the skeleton of Model 5.

Bootstrap Confidence Interval		Period
Lower Limit	Upper Limit	
3.184	3.218	8
3.219	3.273	9
3.274	3.312	8
3.313	3.363	9
3.364	3.405	10
3.406	3.431	11
3.432	3.445	12
3.446	3.451	13
3.452	3.452	14
3.453	3.461	1

DISCUSSION

Examination of Tables 1–7 drew the following conclusion:

1). The most frequent limit cycles that occur are the original limit cycle of the models that has a period of 9 and limit cycles with periods of a multiple of 9 or with periods close to 9.

2). The two models appear to retain the original limit cycle of period 9 in the neighbourhood of each true parameter value, which suggests that the original limit cycle of each one of the two models is robust.

3). Model 5 appears to be more sensitive to minor changes in some parameters than others.

4). Where the same limit cycle is exhibited among the sub-intervals within the bootstrap confidence interval of each parameter, the ones where the original limit cycle occurred are the largest.

CONCLUSION

It should be noted that these results are specific to Model 5. Moreover, we have used only one initial value in simulated data from the skeletons of Model 5, which does not guarantee the stability of the limit cycles found during the experiment.

For future work, we suggest using more models with different initial values before these results can be generalized. In addition, it would be interesting to repeat this study using skeletons of bootstrap models rather than skeletons of initial models like the ones adopted in our experiment. In other words, it would be attractive to see if fitted, a SETAR model gave rise to a limit cycle behaviour of a specific period, whether the corresponding bootstrap model would give rise to the same limit cycle, and how sensitive it is to small changes in the bootstrap model parameters.

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