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Advancing Neutrosophic Topology through b-Open Sets and their Application to Neutrosophic b-Compact Spaces

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Abstract

The main objective of this study is to develop a neutrosophic topology by applying topological mechanisms to establish an induced neutrosophic topology and new topological spaces known as neutrosophic subspaces. In addition, we apply specific topological tools to neutrosophic concepts (open neutrosophic sets of type b) and study the properties of this type of neutrosophic openness. The resulting space, which does not satisfy the three topological conditions but is limited to a supra topology, is called a neutrosophic b supra topological space, and we investigate its main properties. Based on this framework, the concept of b neutrosophic compactness is developed as a natural generalization of compactness in neutrosophic spaces. We establish that every b-compact neutrosophic space is a compact neutrosophic space, but the converse is not true. The research also examines how b-compactness behaves under various mappings, such as continuous and open neutrosophic functions, such as connected and open neutrosophic functions. Furthermore, the research also shows that every neutrosophic supra b-compact space is neutrosophic supra compact and neutrosophic b-compact. The results show that neutrosophic b-compactness serves as a flexible and powerful tool capable of describing and explaining real-world phenomena that are predominantly characterized by uncertainty. Accordingly, this approach reinforces the foundations of neutrosophic topology, which increases its applicability in data analysis, leading to decisions that are more precise.

Keywords: Neutrosophic B-Open Sets, Neutrosophic B-Closed Sets, Neutrosophic Subspaces, Neutrosophic B-Continuous Functions, Neutrosophic B-Compact Spaces.

INTRODUCTION

Despite the great success of Zadeh's fuzzy set theory (Zadeh, 1965) and the subsequent intuitionistic fuzzy set (IFS) theory introduced by (Atanassov, 1986), which added degrees of non-membership, these theories are still unable to describe vague and contradictory information and handle real-world phenomena, despite being widely applied in data mining, pattern recognition, information retrieval, decision making, machine learning, image processing, and related fields. To deal with uncertainty, vague, incomplete, and contradictory information, (Smarandache, 1998) introduced "Neutrosophy: Neutrosophic Probability, Set, and Logic." Subsequently, he proposed the concept of the neutrosophic set (Smarandache, 1999). The neutrosophic set is considered a powerful general formal framework that generalizes the concepts of classical sets, fuzzy sets, and intuitionistic fuzzy sets. A neutrosophic set is characterized by three independ-

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ent functions: truth-membership (T), indeterminacy-membership (I), and falsity-membership (F), each residing in the non-standard unit interval] $^{-}0$, 1^{+} [.

Topology is distinguished by its remarkable ability to understand the properties of phenomena in an abstract way. The integration of neutrosophy with this science has led to the emergence of neutrosophic topology, whose applications are clearly evident in various fields.

In 2012, (A. Salama & S. Alblowi, 2012; A. Salama & S. A. Alblowi, 2012) were instrumental in laying the foundational axioms of neutrosophic topological spaces (NTSs), defining neutrosophic open sets, closure, and interior operators. They introduced the neutrosophic topological space as a generalization of fuzzy and intuitionistic fuzzy topological spaces. Furthermore, (Salama, Smarandache, & Kromov, 2014) studied neutrosophic closed sets and neutrosophic continuous functions in neutrosophic topological spaces. In the same year, (Salama, Smarandache, & Alblowi, 2014) introduced new neutrosophic crisp topological concepts. Exploring the various generalized forms of open sets is a central theme in neutrosophic topology, aiming to develop topological structures supported by more precise and general tools. (Al Amin, 2022) introduced the concept of neutrosophic pre-approximation space, defined neutrosophic pre-rough sets through pre-open sets, and studied their fundamental properties. (Iswarya & Bageerathi, 2016) studied the concept of neutrosophic semi-open sets and neutrosophic semi-closed sets. (Imran et al., 2017) introduced neutrosophic semi-α open sets and studied their fundamental properties. (Arokiarani et al., 2017) defined neutrosophic semi-open (resp. pre-open and α-open) functions and investigated their relations. The concept of b-open sets, initially defined by (Andrijević, 1996) in general topology, was also used and later extended to fuzzy and intuitionistic fuzzy topology, and finally to neutrosophic topology.

In neutrosophic theory, a neutrosophic b-open set (NBOS) is defined as a set contained in the union of the neutrosophic interior of its closure and the neutrosophic closure of its interior. This class of sets, being strictly larger than the family of neutrosophic semi-open and neutrosophic pre-open sets, offers a more flexible and comprehensive view of "near-openness." Many researchers, such as (Arokiarani et al., 2017), have focused on studying partial neutrosophic topological systems (NBOSs) and related concepts, such as b-type continuous neutrosophic functions and b-type neutrosophic closure, while examining their fundamental properties and mutual relationships. Ebenanjar, (Ebenanjar et al., 2018) studied neutrosophic b-open sets in a neutrosophic topological space.

The notion of compactness in topological spaces was introduced by (Gupta & Gaur, 2018; Roy & Mukherjee, 2007) introduced the notion of C-compactness in topological spaces via grills. In addition, many researchers have contributed to compactness in neutrosophic topological spaces (Das¹ & Tripathy, 2020; Das et al., 2020; Pal et al., 2022; Singal & Singal, 1968) . Following their works, (Dhar, 2023) introduced compactness and neutrosophic topological spaces via grills. (Dey & Ray, 2023) defined and studied neutrosophic local pre-compactness. The idea of neutrosophic b-compactness stacking stands out as an effective and fundamental generalization of classical compactness within the neutrosophic framework. This modern concept is characterized by a wider application range than standard neutrosophic stacking, as it uses a wider class of open sets, making it a more accurate and flexible property.

The main objective of this study is to deepen the theoretical framework of neutrosophic topology by formulating and investigating neutrosophic b-compact spaces. These spaces provide a clear generalization of the concept of compactness, enabling more comprehensive tools for analyzing ambiguity-affected systems. Through a series of theories, properties, and examples, we elucidate the relationships between b-compactness and other topological properties, including subspaces and supra spaces, and we examine the effects of certain types of neutrosophic functions. Finally, we demonstrate that the study of neutrosophic compactness in particular topological types provides a solid theoretical foundation for future applications of neutrosophic topology in various uncertain domains.

Preliminaries:

Definition 1 (Andrijević, 1996)

Let X be a topological space. A subset $A \subseteq X$ is said to be b-open if $A \subseteq cl\,(int\,(A)) \cup int\,(cl\,(A))$. The complement of a b-open set is called b-closed.

Definition 2 (Andrijević, 1996)

The b-closure of a set A, denoted by $bcl(A)$, is the intersection of all b-closed sets containing A. It is the smallest b-closed set containing A.

The b-interior of a set A denoted by $bint(A)$, is the union of all b-open sets contained in A. It is the largest b-open set contained in A.

The family of all b-open (resp. b-closed) sets in a space X will be denoted by $BO(X)$ (resp. $BC(X)$).

Proposition 2.1 (Andrijević, 1996)

i. The union of any family of b-open sets is a b-open set.
ii. The intersection of an open and a b-open set is a b-open set.

Definition 3

The b-boundary of a set A of a space X (resp. $b-bd(A)$) is defined by $b-bd(A) = bcl(A) - bint(A) = bCl(A) - bCl(A^C)$.

Definition 4 A subset $N \subseteq X$ is called a b-neighborhood (briefly b-nbd) of a point $x \in X$ if there exists a b-open set $U \subseteq N$ such that $x \in U \subseteq N$.

Definition 5 (Smarandache, 1998; Smarandache, 1999)

Let X be a non-empty set of points. A neutrosophic set A in X (NS for short) is characterized by a truth membership function μ, an indeterminacy membership function σ and a falsity membership function γ, where $\mu_A(x)$, $\sigma_A(x)$ and $\nu_A(x)$ are real standard or nonstandard subsets of $[0,1]$. It can be written as $A = \{< x: \mu_A(x), \sigma_A(x),\ \nu_A(x) >, x \in X\}$. There is no restriction on the sum of $\mu(x)$, $\sigma(x)$ and $\gamma(x)$.

$0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3$.

Definition 6 (Smarandache, 1998; Smarandache, 1999)

Let X be a non-empty set of points. Let A and B are NSs of the form $A = < \mu_A(x), \sigma_A(x), \nu_A(x) : x \in X >$ and $B = < \mu_B(x), \sigma_B(x), \nu_B(x) : x \in X >$. Then:

i. $A \subseteq B$ *if and only if* $\mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$
ii. $A^C = < \nu_A(x), \sigma_A(x), \mu_A(x) : x \in X >$.
iii. $A \cap B = < \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x) : x \in X >$.

iv. $A \cup B = < \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_A(x), \nu_A(x) \wedge \nu_B(x) : x \in X >$.

v. $A - B = A \cap B^C$

Neutrosophic Topological Spaces:

Definition 1 (A. Salama & S. A. Alblowi, 2012)

A neutrosophic topology (NT for short) is a family τ_N of neutrosophic subsets in X satisfying the following axioms:

i. $0_N, 1_N \in \tau_N$, where $0_N = \{< x, 0,0,1 >\}, 1_N = \{< x, 1,1,0 >\}, x \in X$.

ii. $A \cap B \in \tau_N$ for any $A, B \in \tau_N$.

iii. $\cup Ai \in \tau_N$ for any arbitrary family $\{Ai : i \in J\} \subseteq \tau_N$.

In this case the pair (X, τ_N) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in τ_N is known as a neutrosophic open set (NOS for short) in X.

Definition 2 (A. Salama & S. A. Alblowi, 2012)

A neutrosophic set F is neutrosophic closed (NCS for short) if and only if it $(F)^c$ is NOS.

Definition 3 (A. Salama & S. A. Alblowi, 2012)

Let A be an NS in NTS (X, τ_N) . Then

$N.int(A) = \cup \{G: G \text{ is an NOS in } X \text{ and } G \subseteq A\}$ is called a neutrosophic interior of A;

$N.cl(A) = \cap \{G: G \text{ is an NCS in } X \text{ and } G \supseteq A\}$ is called a neutrosophic closure of A.

Proposition 3.1

For any neutrosophic set A in (X, τ_N) we have:

i. A is an NCS in X if $N.cl(A) = A$.

ii. A is an NOS in X if $N.int(A) = A$.

iii. $N.cl(A^C) = (N.int(A))^C$.

iv. $N.int(A^C) = (N.cl(A))^C$.

Definition 4

Let (X, τ_N) be an NTS. Neutrosophic boundary of a subset $A \subseteq X$ is the difference between neutrosophic closure and neutrosophic interior of A and denote it by $N.bd(A) = N.cl(A) - N.int(A)$

Example 1 Let X={a ,b}. Consider the neutrosophic sets on X; A ={<a,0.2,0.2,0.7>, <b, 0.3, 0.3, 0.6>}, B = {<a, 0.6, 0.5, 0.4>, <b,0.6,0.7,0.1>} and C={<a,0.5,0.5,0.4>, <b,0.6,0.6,0.4>}.

Let $\tau_N = \{0_N, 1_N, C\}$ be a neutrosophic topology on X. Then (X, τ_N) is neutrosophic topological space, C is NOS and C^C is NCS in X. Also, we have $N.int(A) = 0_N, N.cl(A) = 1_N$, $N.bd(A) = 1_N$. Also $N.int(B) = C, N.cl(A) = C^C$. $N.bd(B) = 0_N$.

Definition 5 (Salama, Smarandache, & Kromov, 2014)

Let $f : (X, \tau_N) \to (Y, \theta_N)$ be a function, then we have:

i. f is called neutrosophic continuous (briefly N-continuous) if the inverse image of every NOS in Y, is NOS in (X, τ_N)

ii. f is called neutrosophic open if the image of every NOS in X, is NOS in (Y, θ_N).

iii. f is called neutrosophic closed if the image of every NCS in , is NCS in (Y, θ_N).

iv. f is called a neutrosophic homeomorphism if f is bijective and f, f^{-1} are N-continuous.

Definition 6 (Jayaparthasarathy et al., 2019)
Let X be a non-empty set. A subcollection $\tau_N^* \subseteq (X, \tau_N)$ is said to be a neutrosophic supra topology if the sets $0_N, 1_N$ belong to τ_N^* and τ_N^* is closed under arbitrary union. Then the ordered pair (X, τ_N^*) is called a neutrosophic supra topological space on X.

If $A \in \tau_N^*$ then A is called a neutrosophic supra open set (NSOS) and its complement A^C is called a neutrosophic supra closed set (NSCS) in (X, τ_N^*).

Neutrosophic Induced Topology and Neutrosophic Subspaces:
New neutrosophic spaces can be generated from the original space using various topological techniques. By intersecting a subset of the space with all open sets in the original topology, we can generate an induced topology and a neutrosophic subspace from the original space. We will explore its most important properties.

Definition 1

Let (X,τ_N) be an NTS and $W \subseteq X$. We define the family τ_{N_W} as a family of subsets of W, as follows:
$\tau_{N_W} = \{W \cap U : U \in \tau_N\}$. Note that the elements of τ_{N_W} are intersections W with every open set in X. Members of τ_{N_W} are called τ_{N_W}-open sets in W and their complements are called τ_{N_W} -closed set in W.

Theorem 1

Let (X,τ_N) be an NTS and $W \subseteq X$. Then τ_{N_W} is a topology and it is called neutrosophic induced topology on W.

Proof: Utilizing definition of τ_{N_W}, the proof becomes evident.

Definition 2

Let (X,τ_N) be an NTS, then the pair (W,τ_{N_W}) is called a neutrosophic topological subspace of (X, τ_N). The elements of τ_{N_W} are called τ_{N_W} open sets and their complements are called τ_{N_W} closed sets.

Theorem 2

Let (X,τ_N) be an NTS and (W,τ_{N_W}) be a neutrosophic topological subspace of X. If $W \in \tau_N$, then τ_{N_W} is a subfamily of τ_N i.e., If W is an open set in (X,τ_N), then every open set in (W,τ_{N_W}) is open in (X,τ_N).

Proof: Let $V \in \tau_{N_W}$ then there exist $U \in \tau_N$ such that $V = W \cap U$, but $W \in \tau_N$ (by hypothesis), also $U \in \tau_N$, then $W \cap U \in \tau_N$. Therefore $V \in \tau_N$.

Theorem 3

If K is a neutrosophic subspace of W and W is a neutrosophic subspace of X, then K is a neutrosophic subspace of X.

Proof: Let (X, τ_N) be an NTS and $W \subseteq X, K \subseteq W$, to prove K is a neutrosophic subspace of X. First, since $K \subseteq W \subseteq X$, then $K \subseteq X$. Let $B \in \tau_{N_K}$ then there exist $V \in \tau_{N_W}$; $B = K \cap V$, since $V \in \tau_{N_W}$ so there exist $U \in \tau_N$; $V = W \cap U$.

Now, $B = K \cap V \Rightarrow B = K \cap (W \cap U) = (K \cap W) \cap U = K \cap U$. Therefore, K is a neutrosophic subspace of X.

Remark 1

From the definition of the induced topology τ_{N_W}, we note that
$V \in \tau_{N_W} \Leftrightarrow \exists U \in \tau_N ; V = W \cap U$.
As for closed sets, the previous statement satisfies such that:
$F_W \in (\tau_{N_W})^c \Leftrightarrow \exists F \in (\tau_N)^c ; F_W = W \cap F$. Which is equivalent to $F_W \in \mathcal{F}_{N_W} \Leftrightarrow \exists F \in \mathcal{F}_N ; F_W = W \cap F$.
Where $\mathcal{F}_N$ the family of closed sets in X and $\mathcal{F}_{N_W}$ is the family of closed sets in W.

Proposition 1

Let (W, τ_{N_W}) be a neutrosophic subspace of an NTS (X, τ_N), then an NCS F_W of W is NC in X if and only if W is NC in X.
Proof: It is obvious.

Theorem 4

Let (X, τ_N) be an NTS and (W, τ_{N_W}) be a neutrosophic subspace topology of X and $A \subseteq W \subseteq X$, then we have:
i. $\bar{A}_{\tau_{N_W}} = \bar{A}_{\tau_N} \cap W$.
ii. $W \cap A^0{}_{\tau_N} \subseteq A^0{}_{\tau_{N_W}}$.
iii. $W \cap b(A)_{\tau_N} \supseteq b(A)_{\tau_{N_W}}$.
Proof: This follows directly by applying the definition of neutrosophic subspace topology.

Neutrosophic b-open sets and their properties
In this section, we study some characterizations and properties of neutrosophic b-open and neutrosophic b-closed sets.

Definition 1 (Al Amin, 2022)
Let A=$< \mu_A(x), \sigma_A(x), \nu_A(x) : x \in X >$ be an NS in (X, τ_N). Then:
i- A is neutrosophic pre-open set if and only if $A \subseteq Ncl\,(Nint(A))$.
ii- A is neutrosophic pre-closed set if and only if $cl(Nint(A)) \subseteq A$.

Definition 2: Let (X, τ_N) be an NTS and a subset $A \subseteq X$ is called
i. A neutrosophic pre-open set (El Amin, 2022) if $A \subseteq Nint\,(Ncl(A))$ and a neutrosophic pre-closed set if $Ncl\big(Nint(A)\big) \subseteq A$.
ii. A neutrosophic semi-open set (Iswarya & Bageerathi, 2016) if $A \subseteq Ncl\,(Nint(A))$ and a neutrosophic semi-closed set if $Nint\big(Ncl(A)\big) \subseteq A$.

Definition 3 (Arokiarani et al., 2017; Ebenanjar et al., 2018)
A neutrosophic set A in an NTS (X, τ_N) is called:
i. A neutrosophic b-open (NBO) set iff $A \subseteq Nint\,(Ncl(A)) \cup Ncl\,(Nint(A))$.
ii. A neutrosophic b-closed (NBC) set iff $A \supseteq Nint(Ncl(A)) \cap Ncl(Nint(A))$.
Remark 1
A neutrosophic set is b-open if it can be expressed as a union of a neutrosophic pre-open set and a

neutrosophic semi-open set.

Theorem 1

Let (X,τ_N) be an NTS, every neutrosophic pre-open set and every neutrosophic semi-open set are neutrosophic b-open sets.

Proof: This follows directly from Definitions 5.2 and 5.3.

Theorem 2

For an NS A in an NTS (X,τ_N), we have:

i. A is an NBO set iff A^c is an NBC set.

ii. A is an NBC set iff A^c is an NBO set.

Proof: Obvious from Definition 5.3 and Proposition 3.1.

Definition 4

Let (X,τ_N) be an NTS and A be an NS. Then,

i. The neutrosophic b-interior of A denoted by $[Nb.int\ (A)])$ is the union of all neutrosophic b-open sets of (X,τ_N) contained in A.

$Nb.int(A)\ = \cup\{G: G\ is\ a\ NBO\ set\ and\ G\ \subseteq\ A\}$.

ii. The neutrosophic b-closure of A denoted by $[Nb.cl\ (A)]$ is the intersection of all neutrosophic b-closed sets of (X,τ_N) containing A.

$Nb.cl(A)\ = \cap\{H: H\ is\ a\ NBC\ set\ in\ X\ and\ H\ \supseteq\ A\}$.

Clearly $Nb.int(A)$ is the largest neutrosophic b-open set contained in A and $Nb.cl(A)$ is the smallest neutrosophic b-closed set containing A.

Theorem 3

Let A be an NS in an NTS (X,τ_N). Then,

i. $(\mathrm{Nb.int(A)})^{\mathrm{C}} = \mathrm{Nb.cl}(\mathrm{A}^{\mathrm{C}})$.

i. $(\mathrm{Nb.cl(A)})^{\mathrm{C}} = \mathrm{Nb.int}(\mathrm{A}^{\mathrm{C}})$.

Proof: (i) Let A be an NS in NTS (X, τ). Nb.int (A) = = $\cup$\{G: G is an NBO set in X and G ⊆ A\}. Taking the complement and using Theorem 5.2, we get

$(\mathrm{Nb.int(A)})^{\mathrm{C}} = [\cup\{\mathrm{D} :\ \mathrm{D\ is\ a\ NBO\ set\ in\ X\ and\ D} \subseteq\ \mathrm{A}\}]^{\mathrm{C}}$

$= \cap\{\mathrm{D}^{\mathrm{C}}$: D^{C} is an NBC set in X and $\mathrm{A}^{\mathrm{C}} \subseteq \mathrm{D}^{\mathrm{C}}\}$. Replacing D^{C} by M, we get $(\mathrm{Nb.int(A)})^{\mathrm{C}} = \cap = \cap$ \{M: M is an NBC set in X and $\mathrm{M} \supseteq \mathrm{A}^{\mathrm{C}}$\}. This proves $(\mathrm{Nb.int(A)})^{\mathrm{C}} = \mathrm{Nbcl}(\mathrm{A}^{\mathrm{C}})$.

The proof for (ii) is analogous.

Proposition 1

Let (X,τ_N) be an NTS Then:

i. Every neutrosophic open set in (X,τ_N) is a neutrosophic b-open.

ii. Every neutrosophic closed set in (X,τ_N) is a neutrosophic b-closed.

The converse of Lemma 3.1 may not be true in general.

Example 1

Let (X,τ_N) be an NTS where X has only one element, $\tau_N = \{0_N, 1_N\}$ and $A = < 0.5, 0.5, 0.5 >$ be an NS but it is not neutrosophic open nor closed. Now, $Nint(Ncl(A)) \cup Ncl(Nint(A))$ $= < 1{,}0{,}0 > \cup < 1.1{,}0 >=$ $< 1{,}1{,}0 >$, and $A \subseteq < 1{,}1{,}0 >$. So,

$A \subseteq Nint\big(Ncl(A)\big) \cup Ncl\,\big(Nint(A)\big)$, therefore, A is a neutrosophic b-open set. Also, $Nint\big(Ncl(A)\big) \cap Ncl\big(Nint(A)\big) =< 1,0,0 > \cap < 0,1,1 >=< 0,0,1 > \subseteq A$, therefore, A is a neutrosophic b-closed set.

Proposition 2

Let (W, τ_{N_W}) be a neutrosophic subspace of the NTS (X,τ_N) and $A \in N\,(W\,)$, then A is an NBC set in W iff A^c is an NBO set in W.

Proof

Let A be an NBC set in $W \Leftrightarrow [int_W\,(cl_W\,(A))] \cap [cl_W\,(int_W\,(A))] \subseteq A \Leftrightarrow$

$A^c \subseteq [[int_W\,(cl_W\,(A))] \cap [cl_W\,(int_W\,(A))]]^c$

$= [int_W\,(cl_W\,(A))]^c \cup [cl_W\,(int_W\,(A))]^c$

$=[cl_W\,(cl_W\,(A))^c\,] \cup [int_W\,(int_W\,(A))^c\,]$

$= [cl_W\,(int_W(A^c\,))] \cup [int_W\,(cl_W\,(A^c\,))] \Leftrightarrow$

$A^c \subseteq [cl_W(int_W\,(A^c\,))] \cup [int_W\,(cl_W\,(A^c\,))]$.

Therefore A^c is an NBO set in W.

Theorem 4

Let (W, τ_{N_W}) be a neutrosophic subspace of the NTS (X,τ_N). Then:

i. V_W is an NBO set in τ_{N_W} for every NBO set U in τ_N.

ii. F_W is an NBC set in τ_{N_W} for every NBC set F in X.

Proof

i. Let U is an NBO set in X , which implies $U \subseteq [int_X(cl_X(U))] \cup [cl_X(int_X(U))] \Rightarrow$
$V_W \subseteq [(int_X(cl_X(U))) \cup (cl_X(int_X(U)))]_W \subseteq [(int_X(cl_X(U)))\,]_W \cup [\,(cl_X(int_X(U)))]_W$
$\Rightarrow V_W \subseteq [(int_W\,(cl_X(U)\,)_W\,] \cup [(cl_W(int_X(U))_W\,]$
$\Rightarrow V_W \subseteq [(int_W Y(cl_W(U))_W] \cup [cl_W\,(int_W\,(U))_W]$. Therefore, V_W is an NBO set in τ_{N_W} .

ii. The proof is similar to that of (i).

Definition 5

Let (X,τ_N) be an NTS, we define the neutrosophic b-boundary of a subset $A \subseteq X$ as the difference between neutrosophic b-closure and neutrosophic b-interior of A and denote it by $\mathrm{Nb.bd(A)}$, which means that

$Nb.bd(A) = \;Nb.cl(A) - Nb.int(A)$

Proposition 3

Let (X,τ_N) be an NTS. Then for any neutrosophic subsets A and B of X,

i. $Nb.int(A) \subseteq A \subseteq Nb.cl(A)$.

ii. A is an NBO $\Leftrightarrow$ Nb.$i$nt(A) = A.

iii. A is an NBC$\Leftrightarrow$$N$b.cl ($A$) = A.

iii. If $A \subseteq B$ then Nb.$i$nt (A) $\subseteq$ Nb.$i$nt (B) and Nb.cl (A) $\subseteq$ Nb.(B).

iv. $Nb.bd(A) = \;Nb.cl(A) \cap Nb.cl(A^c)$

Proof: It is obvious.

Theorem 5

Let (X,τ_N) be an NTS and $A \subseteq X$ be a NS. Then we have $N.bd(A) \subseteq Nb.bd(A)$

Proof

Assume that $\mathrm{N.bd(A)} \nsubseteq \mathrm{Nb.bd(A)}$. $x \in N.bd(A)$. However $x \notin Nb.bd(A)$, which means that $x \notin Nb.cl(A)$ or $x \notin Nb.cl(A^c)$. Let us $x \notin Nb.cl(A)$ implies there exist b-closed set $F, A \subseteq F\ and\ x \notin F$, then F^c is a b-open set and $x \in F^c$. Since $A \subseteq F$ then $F^c \subseteq A^c$, so there exist b-open set F^c s.t $F^c \cap A = \emptyset$ which contradiction with $x \in Nb.cl(A)$, in the same way we conclude that $x \in Nb.cl(A^c)$. Then $x \in Nb.cl(A) \cap Nb.cl(A^c) = Nb.bd(A)$. so, if $x \in \mathrm{N.bd(A)}$ then $x \in \mathrm{Nb.bd(A)}$. Therefore $\mathrm{N.bd(A)} \subseteq \mathrm{Nb.bd(A)}$.

The converse of the previous theorem is generally not true, and finding an illustrative example remains an open question for future research.

Proposition 4

Let (X,τ_N) be an NTS. Then:

i. The arbitrary union of neutrosophic b-open sets is a neutrosophic b-open set.
ii. The arbitrary intersection of neutrosophic b-closed sets is a neutrosophic b-closed set.

Proof

Let $\{A_i\}_{i\in I}$ be a family of N-bO sets and $A = \cup_{i\in I} A_i$, we want to show $A \subseteq Nint\big(Ncl(A)\big) \cup Ncl\big(Nint(A)\big)$. Then, for each $i \in I$, we have $A_i \subseteq Nint\big(Ncl(A_i)\big) \cup Ncl\big(Nint(A_i)\big)$, since $A_i \subseteq A$ for all i. Taking the union over all $i \in I$, we get

$= \cup_{i\in I} A_i \subseteq \cup[Nint\big(Ncl(A_i)\big) \cup Ncl\big(Nint(A_i)\big)]$

$= Nint[\cup\big(Ncl(A_i)\big)] \cup Ncl[\cup\big(Nint(A_i)\big)]$

$= Nint(Ncl[\cup(A_i)]) \cup Ncl(Nint \cup (A_i))$. Hence $A = \cup A_i$ is a neutrosophic b-open set. The second part follows immediately from the first part by taking complements.

Remark 2

The intersection of two neutrosophic b-open sets may not be neutrosophic b-open. In addition, the union of two N-*b*-closed sets may not be an N-*b*-closed set. This is proved as follows:

Example 2

Let (X,τ_N) be an NTS and $A, B \subseteq X$, where $X = \{a, b\}, \tau_N = \{0N, 1N, A, B\}$

$A = \{< a, 0.5, 0.2, 0.4), (b, 0.6, 0.1, 0.3) >\}$, $B = \{< a, 0.3, 0.5, 0.6), (b, 0.4, 0.4, 0.5) >\}$ are NBO sets, $A \cap B = \{< a, 0.3, 0.2, 0.6), (b, 0.4, 0.1, 0.5) >\}$ is not NBO.

Also, the sets $E = \{(a, 0.6, 0.5, 0.6), (b, 0.5, 0.6, 0.7)\}$, $F = \{(a, 1, 0, 1), (b, 0.9, 0.1, 0.1)\}$ are two NBC sets but their union $E \cup F = \{(a, 1, 0, 0.6), (b, 0.9, 0.1, 0.1)\}$ is not an NBC.

Remark 3: For a neutrosophic topological subspace (X,τ_N), we denote the family of all neutrosophic b-open sets by $NBO(X, \tau_N)$.

Theorem 6

Let (X,τ_N) be an NTS and $A, B \subseteq X$ be an NS. The following are true:

i. $Nb.int(A \cap B) = Nb.int(A) \cap N\,b.int(A)$

ii. $Nb.cl(A \cap B) \subseteq Nb.cl(A) \cap Nb.cl(A)$

iii. $Nb.int(A \cup B) \supseteq Nb.int(A) \cup N\,b.int(A)$

iv. $Nb.cl(A \cup B) = Nb.cl(A) \cup Nb.cl(A)$

Neutrosophic b supra topological spaces:

From the previous section, we note that the arbitrary union of neutrosophic b-open sets is a neutrosophic b-open, while the intersection of two neutrosophic b-open sets may not be neutrosophic b-open. This means that the family of all neutrosophic b-open sets does not generally form a topology. However, we can develop the concept of $NBO(X,\tau_N)$ into new spaces, which are called neutrosophic b supra topological spaces.

Theorem 1

Let (X,τ_N) be a NTS, then $NBO(X,\tau_N)$ is a supra topology on (X,τ_N) and denoted by (X,τ_N^{bS})

Proof: We verify the axioms of a supra topology:

i. It is clear that $0_N, 1_N \in \tau_N$

ii. Follows directly from Proposition 4.3(i).

Definition 1

Let (X,τ_N^{bS}) be an NSTS, $A \subseteq X$ is called:

i. A neutrosophic supra b-open set (NSBO) iff $A \subseteq N^S_{b.int}(N^S_{bcl}(A)) \cup N^S_{bcl}(N^S_{b.int}(A))$.

ii. A neutrosophic supra b-closed set (NSBC) iff $A \supseteq N^S_{b.int}(N^S_{bcl}(A)) \cap N^S_{bcl}(N^S_{b.int}(A))$.

Remark 1

A neutrosophic supra set is b-open if it can be expressed as a union of a neutrosophic supra pre-open set and a neutrosophic supra semi-open set.

Theorem 2

For an NS A in an NTS (X,τ_N), we have:

i. A is an NSBO set iff A^c is an NSBC set.

ii. A is an NSBC set iff A^c is an NSBO set.

Proof

The proof is analogous to that of Theorem 5.2, replacing NBO and NBC with NSBO and NSBC, respectively.

Definition 2

Let (X,τ_N^{bS}) be an NSTS. Then, the neutrosophic supra b interior (in short $N^S_{b.int}$) and neutrosophic supra b closure (in short N^S_{bcl}) of an NS A are defined by:

$N^S_{b.int}(A) = \cup\{U : U \text{ is an NSBO set in} X \text{ and } U \subseteq A\}$.

$N^S_{bcl}(A) = \cap\{V : V \text{ is an NSBCset in } X \text{ and } A \subseteq V\}$.

Clearly $N^S_{b.int}(A)$ is the largest neutrosophic supra b-open set contained in A and $N^S_{bcl}(A)$ is the smallest neutrosophic supra b-closed set containing A.

Theorem 3

Let (X,τ_N^{bS}) be an NSTS, $A \subseteq X$. Then.

i. $(N^S_{b.int}(A))^C = N^S_{bcl}(A^C)$.

ii. $(N^S_{bcl}(A))^C = N^S_{b.int}(A^C)$.

Proof

i. By Definition 6.2, $N^S_{b.int}(A) = \cup\{U : U \text{ is an NSBO set in} X \text{ and } U \subseteq A\}$. Taking complements

and using Theorem 6.2, we get $(N^S_{b.int}(A))^C$ =[$\cup$\{U : U is an NSBO set inX and U $\subseteq A$\}]^C
=$\cap$\{U^C: U^C is an NSBC set in X and $A^C \subseteq U^C$\}..
Replacing U^C by V, we get $(N^S_{b.int}(A))^C$=$\cap$\{V: V is an NSBC set in X and $V \supseteq A^C$\}, This proves $(N^S_{b.int}(A))^C = N^S_{bcl}(A^C)$.
The proof for (ii) is analogous.

Proposition 1

Let (X, τ_N^{bS}) be an NSTS, $A \subseteq X$. Then

i. Every neutrosophic supra open set is a neutrosophic supra b-open.
ii. Every neutrosophic supra closed set is a neutrosophic supra b-closed.

Proof

The proof follows directly from Definition 6.1 and the fact that every supra open set is NSBO.

Theorem 4

Let (X, τ_N^{bS}) be an NSTS, $A.B \subseteq X$. Then the following properties hold:

i. A is NSBO if and only if A= $N^S_{b.int}(A)$.
ii. A is NSBC if and only if $A = N^S_{bcl}(A)$.
iii. if $A \subseteq B$ then $N^S_{b.int}(A) \subseteq N^S_{b.int}(B)$, $N^S_{bcl}(A) \subseteq N^S_{bcl}(B)$.
iv. $N^S_{b.int}(A) \cup N^S_{b.int}(B) \subseteq N^S_{b.int}(A \cup B)$.
v. $N^S_{b.int}(A) \cap N^S_{b.int}(B) \supseteq N^S_{b.int}(A \cap B)$.
vi. $N^S_{bcl}(A) \cup N^S_{bcl}(B) \subseteq N^S_{bcl}(A \cup B)$.
vii. $N^S_{bcl}(A) \cap N^S_{bcl}(B) \supseteq N^S_{bcl}(A \cap B)$.

Proof

We prove some parts; the remaining parts follow similarly.

iii. Let$A \subseteq B$. By Definition 6.2, $N^S_{b.int}(A) = \cup$\{U : U is an NSBO set inX and U $\subseteq A$\}
$\subseteq \cup$\{U : U is an NSBO set inX and U $\subseteq B$\}
$= N^S_{b.int}(B)$.

iv. Since $A, B \subseteq A \cup B$ then we have:
$N^S_{b.int}(A) \cup N^S_{b.int}(B) \subseteq N^S_{b.int}(A \cup B)$
$N^S_{bcl}(A) \cup N^S_{bcl}(B) \subseteq N^S_{bcl}(A \cup B)$.

v. Since $A \cap B \subseteq A, B$ then we have:
$N^S_{b.int}(A) \cap N^S_{b.int}(B) \supseteq N^S_{b.int}(A \cap B)$
$N^S_{bcl}(A) \cap N^S_{bcl}(B) \supseteq N^S_{bcl}(A \cap B)$.

Remark 1

As we have noticed in Theorem 5.3, in the case of NBO sets in neutrosophic topological space, we have:
$Nb.int(A \cap B) = Nb.int(A) \cap N\, b.int(A)$
And $Nb.cl(A \cup B) = Nb.cl(A) \cup Nb.cl(A)$
However, these equalities are not necessarily true in neutrosophic supra topological spaces. The following examples illustrate this remark.

Example 1

Let (X, τ_N^{bS})be an NSTS, where $X = \{p, q, s\}$ with neutrosophic topology $\tau_N^{bS} =$

$\{0_N, 1_N, ((0.5,1,0), (0.5,1,0), (0.5,0,1)), ((0.25,0,1), (0.25,0,1), (0.75,1,0)), ((0.5,1,1), (0.5,1,1), (0.5,0,0)\}$

Let $A_1 = \{(0.5,1,0.25), (0.5,1,0.25), (0.5,0,0.75)\}$, $B_1 = \{(0.5,0.5,1),(0.5,0.5,1),(0.5,0.5,0)\}$.

$A_2 = \{(0.5, 0.5, 0), (0.5, 0.5, 0), (0.5, 0.5, 1)\}$, $B_2 = \{(0.5,0,0.5), (0.5,0,0.5), (0.5,1,0.5)\}$.

Then $N^S_{b.int}(A_1) = \{(0.5,1,0), (0.5,1,0), (0.5,0,1)\}$ and

$N^S_{b.int}(B_1) = \{(0.25,0,1), (0.25,0,1), (0.75,1,0)\}$. So

$N^S_{b.int}(A_1) \cap N^S_{b.int}(B_1) = \{((0.25,0,0), (0.25,0,0), (0.75,1,1)\}$, but

$A_1 \cap B_1 = \{(0.5,0.5,0.25), (0.5,0.5,0.25), (0.5, 0.5, 0.75))\}$.

Thus $\boldsymbol{N^S_{b.int}(A_1 \cap B_1) = \{(0,0,0), (0,0,0), (1,1,1)\} = 0_N}$.

Therefore $N^S_{b.int}(A_1 \cap B_1) \neq N^S_{b.int}(A_1) \cap N^S_{b.int}(B_1)$.

Now, for the second relation, we have:

$N^S_{bcl}(A_2) = ((0.75, 1, 0), (0.75,1,0), (0.25,0,1))$,

$N^S_{bcl}(B_2) = ((0.5,0,1), (0.5,0,1), (0.5,1,0))$.

So $N^S_{bcl}(A_2) \cup N^S_{bcl}(B_2) = ((0.75, 1, 1), (0.75,1,1), (0.25,0,0))$. Also,

$A_2 \cup B_2 = ((0.5,0.5,0.5), (0.5,0.5,0.5), (0.5,0.5,0.5))$ and

$\boldsymbol{N^S_{bcl}(A_2 \cup B_2) = ((1,1,1), (1,1,1), (0,0,0)) = 1_N}$.

Therefore $N^S_{bcl}(A_2 \cup B_2) \neq N^S_{bcl}(A_2) \cup N^S_{bcl}(B_2)$.

Theorem 5

Let (X, τ^{bS}_N) be an NSTS; then every neutrosophic supra b-open set is a neutrosophic b-open set.

Proof: The proof follows directly from definition of neutrosophic supra b-open sets.

Neutrosophic b Compact Spaces:

Neutrosophic b-compactness can be considered a type of neutrosophic compactness, but it imposes additional or different conditions compared to neutrosophic compactness. In this section, we focus on studying the concept of neutrosophic b-compact spaces using neutrosophic b-open sets, and we examine their most important fundamental properties. These properties make neutrosophic b-compactness a powerful tool in neutrosophic topology, especially in practical applications such as decision theory and some complex systems where traditional compactness fails to accommodate and resolve ambiguities in the real world.

Definition 1

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of neutrosophic subsets of the space (X,τ_N). We call $\{A_\alpha\}_{\alpha\in\Lambda}$ a neutrosophic cover of X iff X equals the union of the elements of the family $\{A_\alpha\}_{\alpha\in\Lambda}$, i.e., $X = \bigcup_{\alpha\in\Lambda}\{A_\alpha\}_{\alpha\in\Lambda}$.

Remark 1

If $\{A_\alpha\}_{\alpha\in\Lambda}$ is finite and neutrosophic cover of X, then $\{A_\alpha\}_{\alpha\in\Lambda}$ is called a finite neutrosophic cover of X. If each A_α, $\alpha \in \Lambda$, is a neutrosophic b-open (resp., b-closed) in X and $\{A_\alpha\}_{\alpha\in\Lambda}$ is a neutrosophic cover of X, then it is called a neutrosophic b-open (resp., b-closed) cover of X.

Definition 2

A finite subfamily ${A_i}_{i=1}^{n}$ of a neutrosophic b-open cover $\{A_\alpha\}_{\alpha\in\Lambda}$ on$X$, which is also a neutrosophic b-open cover of$X$, is called a finite neutrosophic subcover.

Definition 3

A neutrosophic topological space (X,τ_N) is said to be neutrosophic b-compact if every neutrosophic b-open cover of X has a finite neutrosophic subcover.

Example 1: let (X,τ_N) be a neutrosophic topological space, $X = \{a, b\}$ and $\tau_N = \{0_N, 1_N, U, V, U \cap V, U \cup V\}$, where

$U = \{< x, (0.8,0.7), (0.2,0.3), (0.1,0.2) >\}$,

$V = \{< x, (0.6,0.9), (0.3,0.1), (0.4,0.05) >\}$.

$U \cap V = \{< x, (0.6,0.7), (0.3,0.3), (0.4,0.2) >\}$

$U \cup V = \{< x, (0.8,0.9), (0.2,0.1), (0.1,0.05) >\}$

Let $C = \{U, V, 1_N\}$ is a neutrosophic b-open cover of$X$, then there exist $S = \{1_N\}$ is a finite neutrosophic subcover. Therefore, (X,τ_N) is a neutrosophic b-compact space.

Definition 4

A subset A of a neutrosophic topological space X is said to be neutrosophic b-compact relative to X if every neutrosophic b-open cover of X has a finite subcover.

Remark 2

Every compact space in the classical sense is neutrosophic b-compact because every open set is also a neutrosophic b-open set. However, the converse is not true. A space can be neutrosophic b-compact without being compact in the classical sense, as neutrosophic b-open sets are more general than standard open sets.

Theorem 1

Every neutrosophic b-compact space is neutrosophic compact.

Proof. Let (X,τ_N) be neutrosophic b compact. Suppose, for contradiction, that X is not neutrosophically compact. Then there exists a neutrosophic open cover $\mathscr{B}$ of X has no finite subcover. By Proposition 5.1, every neutrosophic open set is a neutrosophic b-open, then we have neutrosophic b-open cover $\mathscr{B}$ of X, which has no finite subcover. This is a contradiction to X is neutrosophic b-compact. Hence, X is neutrosophic compact.

Remark 3

The converse of Theorem 7.1 is not true in general as shown by the following example.

Example 2

Let$(\mathbb{N},\tau_N)$ be a neutrosophic topological space, where $N = \{1,2, \dots\}$with $\tau = \{\emptyset_N, \mathbb{N}\}$ For$n \in N$, Clearly$(\mathbb{N},\tau_N)$ is a neutrosophic compact space. Now, Now, for each$n \in N$, define $Gn = \{\langle x, TGn(x), IGn(x), FGn(x) \rangle : x \in N\}$where

$$\begin{cases} TGn(x) = 1, IGn(x) = 0, FGn(x) = 0, & x = n \\ TGn(x) = 0, IGn(x) = 1, FGn(x) = 1, & x \neq n \end{cases}$$

For each $n \in$ N, Gn is an NBO set in $(\mathbb{N},\tau_N)$. Obviously, the collection $C = \{Gn: n \in \text{N}\}$ is a neutrosophic b-open cover of N but it has no finite neutrosophic b-open subcover. Therefore$(\mathbb{N},\tau_N)$ is not a neutrosophic b-compact space. Thus$(\mathbb{N},\tau_N)$ is a neutrosophic compact space but not a neutrosophic b-compact space.

As a special case, the previous theory is true in the case of neutrosophic sets, as shown in the following proposition.

Proposition 1

In an NTS, every neutrosophic b-compact set is a neutrosophic compact set.

Proof: Let $A \subseteq (X, \tau_N)$ be a neutrosophic b-compact set Let $C = \{G_i : i \in \Lambda\}$ be a neutrosophic open cover of A. Since every neutrosophic open set is an NBO set. So G_i is an NBO set for each $i \in \Lambda$. Therefore, C is a neutrosophic b-open cover of A. Since A is b-compact, so there exists a finite subcollection $\{G_{i1}, G_{i2}, \ldots, G_{im}\}$ of C such that $A \subseteq G_{i1} \cup G_{i2} \cup \ldots \cup G_{im}$. Thus, the neutrosophic open cover C of A has a finite neutrosophic open subcover. Hence, A is a neutrosophic compact set.

The following example illustrates that the converse is not true in general.

Example3

Let $X = \{a, b\}$, $A = \{\langle a, 0,1,1\rangle, \langle b, 1,0,0\rangle\}$ is an NS. Let $\tau = \{X, A, \emptyset\}$. Clearly (X, τ_N) is an NTS. Now let $G_n = \left\{\langle a, 0,1,1\rangle, \left\langle b, \frac{n}{n+1}, \frac{1}{n}\frac{1}{n+1}\right\rangle\right\}, n \in N = \{1,2,3, \cdots\}$ is an NBO set for each $n \in$ N. Obviously A is a neutrosophic compact set. We observe that $\{G_n : n \in \text{N}\}$ is a neutrosophic b-open cover of A but it has no neutrosophic open subcover. Therefore, A is not a neutrosophic b-compact set.

Theorem 2

A neutrosophic b-closed subset of a neutrosophic b-compact space X is neutrosophic b-compact relative to X.

Proof

Let A be a neutrosophic b-closed subset of a neutrosophic b-compact space X. Then A^c is neutrosophic b-open in X. Let $\mathscr{B} = \{A_i : i \in \Lambda\}$ be a neutrosophic b-open cover of A. Then $\mathscr{B} \vee \{A^c\}$ is a neutrosophic b-open cover of X. Since X is neutrosophic b-compact, it has a finite subcover say $\{P_1, P_2, \ldots, P_n, A^c\}$. Then $\{P_1, P_2, \ldots, P_n\}$ is a finite neutrosophic b-open cover. Thus, A is neutrosophic b-compact relative to X.

Theorem 3

The image of a neutrosophic b-compact space under a neutrosophic b continuous function is neutrosophic b-compact.

Proof

Let $f : (X, \tau_N) \rightarrow (Y, \theta_N)$ be a neutrosophic b continuous function from a neutrosophic b-compact space (X, τ_N) onto a topological space (Y, θ_N). Let $\mathscr{B} = \{A_i : i \in \Lambda\}$ be a neutrosophic open cover of (Y, θ). Since f is neutrosophic continuous, $\{f^{-1}(A_i) : i \in \Lambda\}$ is a neutrosophic b-open cover of (X, τ). As (X, τ) is neutrosophic b-compact, the neutrosophic open cover $\{f^{-1}(A_i) : i \in \Lambda\}$ of (X, τ) has a finite subcover $\{f^{-1}(A_i) : i = 1, 2, 3\ldots, n\}$. Therefore $X = \cup f^{-1}(A_i)$. Then $f(X) = \cup_{i \in \Lambda} A_i$, that is $Y = \cup_{i \in \Lambda} A_i$. Thus $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\mathscr{B} = \{A_i : i \in \Lambda\}$ for (Y, θ). Hence, (Y, θ) is neutrosophic b-compact.

Theorem 4

Let $f : (X, \tau_N) \rightarrow (Y, \theta_N)$ be a neutrosophic b-open function. If $A \subseteq Y$ is neutrosophic b-compact in Y then $f^{-1}(A)$ is neutrosophic compact in X.

Proof.

Let f: $(X,\tau_N) \to (Y,\theta_N)$ be a neutrosophic b-open, let $\mathscr{B}= \{A_i : i\in\Lambda\}$ be a neutrosophic b-open cover of $f^{-1}(A)$, then $f^{-1}(A) \subseteq \cup_{i\in\Lambda} A_i \Rightarrow A \subseteq f(\cup_{i\in\Lambda} A_i) \Rightarrow A \subseteq \cup_{i\in\Lambda} f(A_i)$, since A_i is an NBO set in (X,τ_N), so $f(A_i)$ is NBO set in (Y,θ_N) for each $i\in I$ as f is an NBO function. Therefore, $C = \{f(A_i): i \in \Lambda\}\}$ is an NPOC of A, since A is neutrosophic b-compact, so C has a finite NPOSC $\{f(A_{i1}), f(A_{i2}), \ldots, f(A_{in})\}$. Therefore $A \subseteq \cup_{i=1}^{n} f(A_{in}) \Rightarrow A \subseteq f(\cup_{i=1}^{n}(A_{in})) \Rightarrow f^{-1}(A) \subseteq \cup_{i=1}^{n}(A_{in})$. Therefore $f^{-1}(A)$ is neutrosophic compact in X.

Theorem 5

Let f: $(X,\tau_N) \to (Y,\theta_N)$ be a neutrosophic open function. If (Y,θ_N) is neutrosophic b-compact then (X,τ_N) is neutrosophic compact.

Proof: This follows directly from Proposition 5.1 and the definition of neutrosophic b-compact.

Proposition 2

Let (X,τ_N) be a neutrosophic topological space. Then:

i. The union of two neutrosophic b-compact sets is neutrosophic b-compact.

i. The intersection of two neutrosophic b-compact sets is neutrosophic b-compact.

Proof

i. Let A and B be two neutrosophic b-compact sets of an NTS (X,τ_N). Let $C = \{Gi : i \in \Lambda\}$ be an arbitrary neutrosophic b-open cover of $A \cup B$. Then $A \cup B \subseteq \cup_{i\in\Lambda} Gi$. Since $A \subseteq A \cup B$, so C is a neutrosophic b-open cover of A. Since A is a neutrosophic b-compact, so there exists a finite subcollection $\{G_{i_1}, G_{i_2}, \ldots, G_{i_m}\}$ *of* C such that $A \subseteq G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_m}$. Also, for a neutrosophic b-compact B, there exists a finite subcollection $\{M_{i_1}, M_{i_2}, \ldots, M_{i_m}\}$ of C such that $B \subseteq M_{i_1} \cup M_{i_2} \cup \ldots \cup M_{i_m}$. Therefore, $A \cup B \subseteq G_{i_1} \cup G_{i_2} \cup \ldots \ldots \cup G_{i_m} \cup M_{i_1} \cup M_{i_2} \cup \ldots \ldots \cup M_{i_m}$. Then $A \cup B$ is a neutrosophic b-compact.

ii. The proof is similar to (i).

Theorem 6

Let (X,τ_N) be a neutrosophic topological space. Then:

i. The intersection of a finite number of neutrosophic b-compact sets is also neutrosophic b-compact.

ii. The union of a finite number of neutrosophic b-compact sets is neutrosophic b-compact.

Proof: This follows immediately from Proposition 7.2 by induction.

Definition 5

Let (X,τ_N) be an NTS. A collection $\{G_i : i \in \Lambda\}$ of neutrosophic sets of X is said to have the finite intersection property (FIP, in short) iff every finite subcollection $\{G_{i_k}: k = 1,2, \ldots n\}$ satisfies the condition $\cap_{i=1}^{n} G_{i_k} \neq \emptyset_N$

Theorem 7

A neutrosophic topological space (X, τ_N) is neutrosophic b-compact if and only if every family of neutrosophic b-closed (NBC) sets with (FIP) has a non-empty intersection.

Proof: $(\Rightarrow)$

Assume (X,τ_N) is neutrosophic b-compact. Let $\{F_i, i \in \Lambda\}$ be a family of N-bC sets with the FIP. We need to show that $\bigcap_{i\in I} F_i \neq \emptyset_N$. Assume, for the sake of contradiction, that $\bigcap_{i\in I} F_i = \emptyset_N$. By De Morgan's laws, $(\bigcap_{i\in\Lambda} F_i)^C = \bigcup_{i\in\Lambda} F_i{}^C = \emptyset_N{}^C = X_N$. Since each F_i is NBC, its complement $F_i{}^C$ is an NBO set. Thus, the family $\{F_i{}^C, i \in \Lambda\}$ is a neutrosophic b-open cover of X. Because X is neutrosophic b-compact, this cover has a finite subcover, say $\{F_{i1}{}^C, F_{i2}{}^C \ldots, F_{in}{}^C\}$. This means $\bigcup_{i=1}^{n} F_i{}^C = X_N$. Taking the complement again, we get $\bigcap_{i=1}^{n} F_i{}^C = X_N{}^C = \emptyset_N$. However, this contradicts. Therefore, our initial assumption was false, and the intersection $\bigcap_{i\in\Lambda} F_i$ must be non-empty.

($\Leftarrow$) Assume that every family of NBC sets with the FIP has a non-empty intersection. Let $\{G_i : i \in \Lambda\}$ be a neutrosophic b-open cover of X. This means $\bigcup_{i\in I} G_i = X_N$. Taking complements, we get $(\bigcup_{i\in I} G_i)^C = \bigcap_{i\in\Lambda} G_i{}^C = X_N{}^C = \emptyset_N$. Since the family $\{G_i{}^C : i \in \Lambda\}$ is an NBC sets with an empty intersection, it cannot have the FIP. This means there must be a finite subfamily $\{G_{i1}{}^C, G_{i2}{}^C, \ldots, G_{in}{}^C\}$ whose intersection is empty. So, $\bigcap_{i=1}^{n} G_i{}^C = \emptyset_N$. Taking complements again, we get $(\bigcap_{i=1}^{n} G_i{}^C)^C = \bigcup_{i\in I} G_i = \emptyset_N{}^C = X_N$. This shows that $\{G_{i1}, G_{i2} \ldots, G_{in}\}$ is a finite subcover of $\{G_i : i \in \Lambda\}$. Therefore, X is neutrosophic b-compact.

Theorem 8

Let (Y,τ_Y) be a neutrosophic subspace of the NTS (X,τ_N), $A \subseteq Y$. If A is neutrosophic b-compact in X, then A is neutrosophic compact in Y.

Proof

Let $\{(G_i)_Y : i \in \Lambda\}$ be an NOC of A, where $G_{iY} \in \tau_Y$ for each $i \in I$. Then $A \subseteq \bigcup_{i\in\Lambda} G_{iY} \Rightarrow A \subseteq \bigcup_{i\in\Lambda} G_i$. Obviously $G_i \in \tau_N$ is an NBO set in X for each $i \in \Lambda$. Therefore, $\{Gi : i \in \Lambda\}$ is an NBOC of A in X. Since A is neutrosophic b-compact in X, there exists a finite subcollection $\{G_{i_k} : k = 1,2,\ldots,n\}$ such that $A \subseteq \bigcup_{k=1}^{n} G_{i_k} \Rightarrow A \subseteq (\bigcup_{k=1}^{n} G_{i_k})_Y \Rightarrow A \subseteq \bigcup_{k=1}^{n} (G_{i_k})_Y$. Therefore, A is neutrosophic compact in X.

Proposition 3

Let (Y,τ_Y) be a neutrosophic subspace of the NTS (X,τ_N) and $A \subseteq Y$. If A is neutrosophic b-compact in Y then A is neutrosophic b-compact in X.

Proof

It is obvious.

Theorem 9

Every neutrosophic supra b-compact space is neutrosophic supra compact.

Proof

Let (X,τ_N) be a neutrosophic supra b-compact space. Therefore, every NSBO-cover of (X,τ_N) has a finite sub-cover. Let (X,τ_N) is not an NS-compact space. Then there exists a neutrosophic b-open cover $\mathscr{B} = \{A_i : i \in \Lambda\}$ of X that has no finite subcover. This contradicts the fact that (X,τ_N) is an NSB compact-space. Therefore, (W,τ) must be an NS compact-space.

Theorem 10

Every neutrosophic supra b-compact space is neutrosophic b-compact.

Proof

Since every neutrosophic supra b-open set is neutrosophic b-open $(NSb - open(X) \subseteq Nb - open(X))$, then every neutrosophic supra b-open cover is a neutrosophic b-open cover.

Let $U = \{Ui: i \in \Lambda, Ui \in Nb - open(X)\}$ be a neutrosophic b-open cover of X *and* $X = \bigcup_{i\in\Lambda} U_i$
Suppose X is neutrosophic supra *b*-compact. Then for every cover by neutrosophic supra b-open sets,

$$X = \bigcup_{j\in J} V_j, V_j\ NSb - open(X)$$

Then there exists a finite subcover $X = \bigcup_{k=1}^{n} V_{jk}$. Consider $\mathbb{V} = \{V_j: j \in J\}$ as a subset of U if applicable. The finite subcover from supra b open sets implies a finite subcover for broader b open covers.
Hence, every neutrosophic supra b-compact space is neutrosophic b-compact.

Remark 4
The converses of Theorems 7.9 and 7.10 are not necessarily true, because the class of neutrosophic b-open sets is larger and may admit covers that do not have finite subcovers when restricted to neutrosophic supra b-open sets.

CONCLUSION

Linking precise topological tools based on flexible concepts with the concept of compactness, provides an important and essential framework that can deal with many relationships in different scientific fields, by introducing the idea of b-open (b-closed) neutrosophic sets, which has been used in studying the concept of neutrosophic b-compactness. We have proved that every neutrosophic b-compact space is a neutrosophic compact space, but the converse is not true. Neutrosophic induced topology and neutrosophic subspaces were established, and their most important properties were studied. This was followed by examining the effects of some neutrosophic functions on spaces and discussing their most important properties, which will help researchers enhance and promote further study of neutrosophic topology and develop a general framework for its applications in practical life.

Future research directions include:

1. Investigating other generalized compactness types in neutrosophic topology, such as neutrosophic b-Lindelöf spaces or countably b-compact spaces.
2. Exploring the connections between b-compactness and separation axioms (such as neutrosophic b-Hausdorff spaces).

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