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Hybrid Triple Quadrature Rule Blending Some Gauss-Type Rules with the classical or the Derivative-Based Newton-Cotes-Type Rules.

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Abstract

Hybrid numerical quadrature rules are widespread techniques for approximate computations of definite integrals. Such hybrid rules combine as many quadrature rules as long as they possess the same degree of precision. The revenue is a new mixed rule with a higher degree of precision than its constituted rules at least by two. Moreover, such mixed rules are quite simple and handy, because they do not involve any extra evaluations of the integrand. That is by relying on the same number of quadrature points of the constituted rules, the acquired hybrid rule performs more efficiently than its ingredients rules. In this paper; a triple hybrid quadrature rule has been constructed for the numerical integration of real definite integrals that do not possess a closed-form anti-derivative. At First, a dual hybrid rule was produced by blending Milne's rule of Newton-Cotes type with the anti-Gaussian quadrature rule to prevail a dual rule of a degree of precision equal to five. Then the acquired dual rule is recombined with the composite derivative-based and mid-Point Newton–Cotes formula producing a hybrid triple rule of degree of precision equal to seven. The accomplished approach is satisfactory and efficient in the approximate evaluation of definite real integrals as confirmed analytically by the error analysis and numerically by some verification examples. To promote the degree of precision of the proposed triple approach, the numerical computations have been implemented in an adaptive environment.

Keywords: Hybrid Quadrature Rule, Milne's Rule, Anti-Gaussian Quadrature, Derivative-Based Newton-Cotes Quadrature Rules, Composite Mid-Point Newton-Cotes Quadrature Rules, Adaptive Quadrature Rule, Numerical Integration.

INTRODUCTION

Numerical quadrature rules have gained great popularity in numerical integration for certain classes of integrals that cannot be integrated analytically. The quadrature rules can be classified either as Newton-Cotes-type or Gauss-type (Atkinson, 2012; Burden and Faires, 2005). The pronounced difference between the two categories of quadrature rules is that the nodes for the Newton-Cotes rules are equally spaced points along the interval of integration. Whereas the Gauss rule picks the nodes differently and does not have to be equally distanced, and the corresponding weights are usually irrational numbers. A great feature of the Gauss-type rule is that the weights are always positive which has a desirable effect on the stability of the quadrature rule. Unfortunately, such a feature cannot be guaranteed for Newton-Cotes rules, especially for a large number of quadrature points. Sermutlu (Sermutlu, 2005) conducted a comparative study between Newton-Cotes-Type and Gauss-Type quadrature rules based on several criteria such as degree of precision, running time, computational cost, coefficient of the leading term of the error, and stability. He claimed that the Gauss-type quadrature rules offer superior performance compared to the Newton-Cotes quadrature rules.

Modified families of closed, Mid-point, and open Newton-Cotes quadrature rules have been recently established by Burg *et al*. (Burg and Degney, 2013; Burg, 2012; Zafar *et al*., 2014), and are known as derivative-based Newton-Cotes formulae. Such formulae require the evaluations of the integrand and its derivative at a smaller number of quadrature points in comparison to the standard Newton-Cotes formulae. Burg *et al*. (Burg and Degney, 2013; Burg, 2012; Zafar *et al*., 2014) state that their modified Newton-Cotes formulae perform considerably well compared with the classical Newton-Cotes (Burg and Degney, 2013; Burg, 2012; Zafar *et al*., 2014; Dehghan *et al*., 2005 a; Dehghan *et al*., 2005b).

Moreover, numerical enhancements of the open and semi-open Newton-Cotes formulae were presented respectively by Dehghan *et al*. (Dehghan *et al*., 2005a; Dehghan *et al*., 2005b). Thus, they claim that the numerically enhanced rules are superior to the classical Newton-Cotes rules of open, closed, and semi-open types. Moreover, a recent approach was first introduced by Das and Pradhan in 1996 for numerical quadrature (Das and Pradhan, 1996). The core idea of their approach is joining a pair of quadrature rules of the same degree of precision to generate a new mixed rule with a better degree of precision. Then several formations of the mixed quadrature rules appeared for numerical computation of real definite integral (Das and Pradhan, 2012; Das and Pradhan, 2013a; Das and Pradhan, 2013b; Jena and Dash, 2011; Tripathy *et. al*., 2015; Patra *et. al.*, 2018). Furthermore, such an approach has been also implemented for the numerical computation of analytic functions (Mohanty, 2010). Such mixed rules have been proven valuable in solving different classes of integral equations either with regular or singular kernels (Jena and Nayak, 2015).

In this paper, a triple hybrid quadrature rule has been formalized for the numerical integration of real definite integrals that do not own closed-form anti-derivatives. The proposed approach blends Milne's rule with the anti-Gauss quadrature rule to generate a dual rule with a degree of precision equal to five. Then the accomplished dual hybrid rule is recombined with the composite derivative-based Newton-Cotes rule to produce a triple hybrid rule of degree of precision equal to seven. The proposed triple hybrid rule will be verified by some integral examples that do not hold an elementary anti-derivative.

The structure of this paper is as follows: The related literature review was reviewed in the introduction section. Then some preliminary concepts were introduced and the notations were used in this paper. In the third section; the dual and the triple hybrid quadrature rules have been established and an error analysis is presented analytically. To verify the acquired approach, some numerical results are shown in the results section followed by a discussion and conclusion.

Preliminaries Concepts

Here some basic definitions need throughout the paper.

Definition 1: An n-point Gaussian-quadrature rule is defined by the formula,

$$I_n(f)=\int_a^b f(x)\,dx \cong \sum_{i=1}^{n} w_i f(x_i)+EI_n(f), \tag{1}$$

where the points x_i are the quadrature points are known as nodes or abscissas, the factors w_i are the corresponding weights and $EI_n(f)$ is the error of the rule (1). The quadrature rule (1) is based on polynomials interpolation. The mechanism of the Gauss quadrature is based on the precision concept, that is the quadrature rule is exact for polynomial of degree less than or equal to $2n-1$. That is the formula (1) exactly integrates first n monomials functions $x^i, i=0,1,2,\ldots,n$. Thus we obtain a non-linear system of moment equations that can be solved yielding the nodes and the corresponding weights.

Definition 2 (Degree of Precision): The degree of precision of a quadrature rule is the highest degree of the polynomial $P_n=x^n$ such that the relevant rule is exact for all the monomials $x^i, i=0,1,2,\ldots,n$. Thus the quadrature rule is exact for all polynomials of degree $\leq n$ and the error does not vanish for $i=n+1, n+2,\ldots$

MATERIALS AND METHODS

Here we outline the definitions of the quadrature rules that were implemented later through the formalization of the hybrid rules either the dual or the triple rules.

Anti-Gaussian Quadrature Rule

Definition 3 (Laurie, 1996): An $(n+1)$- the formula defines point Anti-Gaussian formula,

$$I_{aG(n+1)}(f)=\int_a^b f(x)\,dx \approx \sum_{i=1}^{n+1} w_i f(x_i), \tag{2}$$

where all the weights are positive and the abscissas are real and interlaced by those of the n-point Gaussian formula (1). This rule has $(2n-1)$ as the degree of precision and with the error of the same module but the opposite sign to the error of the n-point Gauss-Legendre quadrature rule (2).

For example, for $n=2$, we have the 3-point anti-Gaussian formula $I_{aG3}(f)$ as,

$$I_{aG3}(f)=\frac{h}{13}\left\{16f(\rho)+5\left[f\left(\rho-h\sqrt{\frac{13}{15}}\right)+f\left(\rho+h\sqrt{\frac{13}{15}}\right)\right]\right\}, \tag{3}$$

where $h=\left(\frac{b-a}{2}\right)$ and throughout the paper $\rho=\left(\frac{a+b}{2}\right)\in[a,b]$ denotes the mid-point of the integration interval. Thus, one has,

$$I_{Exact}(f)=\int_a^b f(x)\,dx = I_{aG3}(f)+E_{aG3}(f), \tag{4}$$

where $E_{aG3}(f)$ is the truncation error of the 3-point anti-Gauss quadrature rule. Thus from equation (4), one has

$$E_{aG3}(f)=I_{Exact}(f)-I_{aG3}(f).$$

This error can be derived by polynomials interpolation (Atkinson, 2012; Burden and Faires, 2005) or by Taylor expansion (Das and Pradhan, 1996) of the functions involved in $I_{aG3}(f)$ about the mid-point ρ of the integration interval $[a,b]$ to yield,

$$E_{aG3}(f) = -\frac{h^5}{135} f^{(4)}(\rho) - \frac{1016\, h^7}{675 \times 7!} f^{(6)}(\rho) - \frac{2144\, h^9}{1125 \times 9!} f^{(8)}(\rho) - \cdots \quad (5)$$

The degree of precision of the 3-point anti-Gauss quadrature rule $I_{aG3}(f)$ is three, and the local truncation error is of the fifth order.

Milne's Rule

Milne's rule is the three-point open Newton-Cotes quadrature rule, and is given by the following formula (Atkinson, 2012; Burden and Faires, 2005)*:*

$$\int_a^b f(x)\,dx \approx I_{Mil}(f) = \frac{4h}{3}\{2[f(a+h) + f(a+3h)] - f(a+2h)\}, \quad (6)$$

where the step-size $h = \left(\frac{b-a}{n+2}\right), n = 2$. Thus, one has,

$$I_{Exact}(f) = \int_a^b f(x)\,dx = I_{Mil}(f) + E_{Mil}(f), \quad (7)$$

where E_{Mil} is the truncation error of the Milne's rule. Thus from equation (7), one has

$$E_{Mil}(f) = I_{Exact}(f) - I_{Mil}(f).$$

This error can be derived by Taylor expansion of the functions involved in $I_{Mil}(f)$ about the mid-point ρ of the integration interval $[a, b]$ to yield,

$$E_{Mil}(f) = \frac{14h^5}{45} f^{(4)}(\rho) + \frac{656h^7}{3 \times 7!} f^{(6)}(\rho) + \frac{976\, h^9}{9!} f^{(8)}(\rho) + \cdots \quad (8)$$

Thus the degree of precision of Milne's rule $I_{Mil}(f)$ is three and the local truncation error is of the fifth order.

Composite Newton-Cotes-Type Derivative-Based and mid-point quadrature rule.

In the current work ،we used a derivative-based quadrature formula that only requires the integrand evaluations at the mid-point of the integration interval $[a, b,]$ and evaluations of odd derivatives at the end-points a and b. Such formula is given as (Burg and Degney, 2013),

$$\int_a^b f(x)\,dx = 2hf(\rho) - \frac{h^2}{6}[f'(a) - f'(b)] + \frac{7h^3}{360}[f'''(a) - f'''(b)] + \frac{62h^7}{3 \times 7!} f^{(6)}(\rho). \quad (9)$$

Thus the degree of precision of this rule is five and the local truncation error is of the seventh order. It should be noted that the weights of the first and third derivatives in equation (9) are of opposite sign. This formula can be put in composite form as (Burg and Degney, 2013).

$$\int_a^b f(x)\,dx = 2h \sum_{k=1}^{N/2} f(x_{2k-1}) - \frac{h^2}{6}[f'(a) - f'(b)] + \frac{7h^3}{360}[f'''(a) - f'''(b)] + \frac{31Nh^7}{3 \times 7!} f^{(6)}(\rho), \quad (10)$$

where the nodes $x_i = a + ih$ and the step-size $h = \left(\frac{b-a}{N}\right)$.

Formula (9) has great features that make it an efficient quadrature rule. For instance, consistently with using the composite formula (10), only derivative evaluations at the end-points are required. This advantageous feature is due to the desirable appearance of the opposite-sign weights of odd derivatives that are involved in the formula (9). Unfortunately, such a handy feature does not persist with the appearance of even derivatives in the composite formula (10) for some values of

N where the weights of derivatives are of the same signs (Burg and Degney, 2013).

Furthermore, the composite formula (10) is progressive concerning the weights of the involved derivatives, that is it can be easily extended to involve high-order odd-derivatives. Therefore to acquire higher accuracy of the formula (10), Burg and Degny (Burg and Degney, 2013) implemented the central difference approximations to the derivatives. Hence the weights for the low-derivative remain unchanged, thus one only needs to compute the weights for the arising derivatives.

Now after we introduce all the quadrature rules that we will use in this paper either of Gauss-type or Newton-Cotes-type, we will formulate our hybrid rules as shown next.

RESULTS AND ERROR ANALYSIS

Here study shows how to formalize the triple hybrid quadrature rule, such formulation has two stages. The first task is to generate the dual rule, and then generate the triple quadrature rule as shown next.

Establishment of the Dual Hybrid Rule Joining the 3-Point Anti-Gauss Rule with Milne's Rule.

Here we show how to mingle two quadrature rules to generate a dual rule of degree of precision seven. The ingredients rules of the hybrid dual rule are the 3-point anti-Gauss rule (3) and Milne's rule (6) both having the same degree of precision equal to five. The core idea of generating the hybrid quadrature rules is to linearly combine the ingredient quadrature rules in such a way that leads to the cancellation of the leading term in the remainder of the ingredient rules as we show next.

To attain a linear combination of the quadrature rules (3) and (6), we multiply equations (4) and (7) respectively by $\frac{1}{15}$ and $\frac{14}{5}$, then add the resulting equations, yielding the following dual hybrid quadrature rule as,

$$I_{DH}(f) = \frac{1}{43}\left[I_{Mil}(f) + 42\, I_{aG3}(f)\right]. \tag{11}$$

where $I_{Mil}(f)$ and $I_{aG3}(f)$ are respectively given by equations (6) and (3).

This error of $I_{DH}(f)$ denoted as $E_{DH}(f)$ can be generated by the following equation,

$$I_{Exact}(f) = I_{DH}(f) + E_{DH}(f). \tag{12}$$

Thus by Taylor expansions of the functions involved in $I_{DH}(f)$ about the mid-point ρ of the integration interval $[a, b]$ one has,

$$E_{DH}(f) = \frac{34976}{3375 \times 7!} h^7 f^{(6)}(\rho) + \frac{335984\, h^9}{5625 \times 9!} f^{(8)}(\rho) + \cdots \tag{13}$$

Hence the degree of precision of the dual hybrid $I_{DH}(f)$ is five and the local truncation error is of ninth order.

Establishment of the Triple Hybrid Rule.

Here study shows how to join the dual hybrid rule I_{DH} given by the equation (11) with the composite quadrature rule (10) of Newton-Cotes-Type and of the same degree of precision of I_{DH} to establish a triple quadrature rule with the degree of precision equal to seven. The formula (10) can be rewritten for $N = 6$ as,

$$\int_a^b f(x)\,dx = 2h[f(x_1)+f(x_3)+f(x_5)] - \frac{h^2}{6}[f'(a)-f'(b)] + \frac{7h^3}{360}[f'''(a)-f'''(b)] + \frac{62h^7}{7!}f^{(6)}(\rho), \qquad (14)$$

where $x_i = a + ih$. This equation can be rewritten as,

$$I_{Exact}(f) = \int_a^b f(x)\,dx = I_{CDM}(f) + E_{CDM}(f), \qquad (15)$$

where E_{CDM} is the truncation error of the composite quadrature rule (14). Given as,

$$E_{CDM}(f) = \frac{62h^7}{7!}f^{(6)}(\rho) + \cdots,$$

By following a similar analogy to the derivation of the dual hybrid rule, we will produce the triple hybrid rule. By a similar analogy of the derivation of the dual hybrid rule, an appropriate linear combination between the dual hybrid rule(11) and the composite quadrature rule (14). Such a linear mixture guarantees the cancellation of the leading term of the remainder of their ingredient rules. Hence, one has,

$$I_{TH}(f) = \frac{1}{31601}[34976\, I_{CDM}(f) - 3375\, I_{DH}(f)], \qquad (16)$$

and the corresponding truncation error of the triple rule I_{TH} is,

$$E_{TH}(f) = \mathcal{O}(h^9). \qquad (17)$$

Hence the degree of precision of the generated triple hybrid $I_{TH}(f)$ is seven and the local truncation error is of ninth order.

Numerical Results

Table (1) shows some integral examples that we consider in this paper with their non-elementary anti-derivative and their approximate values.

Table: (1). Some Integral examples with their non-elementary anti-derivative and their approximate values.

Integral	Exact Value	Approximate Value
$I_1 = \int_{-1}^{1} \frac{e^{e^{\left(\frac{x+3}{2}\right)}}}{2}dx$	$I_1 = Ei(e^2) - Ei(e)$ $\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-y}}{y}dy$	≈ 255.676
$I_2 = \int_1^2 e^{-x^2}\,dx$	$I_2 = \frac{\sqrt{\pi}}{2}[erf(2) - erf(1)]$ $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy$	≈ 0.135257
$I_3 = \int_1^2 \frac{\sin x}{x}dx$	$I_3 = Si(2) - Si(1)$ $Si(x) = \int \frac{\sin x}{x}dx$	≈ 0.6593329906
$I_4 = \int_0^1 \frac{dx}{1+x^4}$	$I_4 = \frac{\pi + 2\coth^{-1}(\sqrt{2})}{4\sqrt{2}}$	≈ 0.86697

Table (2) shows the approximate values of the four integrals I_1, I_2, I_3, and I_4 computed by the dual and the triple hybrid quadrature rules $I_{DH}(f)$ and $I_{TH}(f)$ and their ingredients rules $I_{MI}(f)$, $I_{CDM}(f)$ and $I_{aG3}(f)$. The obtained results have been enhanced to reach a certain degree of precision by implementing an adaptive quadrature algorithm as explained next.

Table: (2). Numerical results computed by the hybrid quadrature rules $I_{HD}(f)$ and $I_{TH}(f)$ compared with its constituent rules I_{MI}, $I_{CDM}(f,)$ and $I_{aG3}(f)$

Integral	$I_{MI}(f)$	$I_{aG3}(f)$	$I_{DH}(f)$	$I_{CDM}(f)$	$I_{TH}(f)$
	Relative Error	Relative Error	Relative Error	Relative Error	Relative Error
I_1	202.8302589	299.7667015	297.51236563	224.5599104	216.76855772
	0.20668986	0.17244816	0.163630999	0.1217007993	0.1521743547
I_2	0.135788265	0.134847269	0.1348691522	0.13532354	0.1353720733
	0.003925903	3.031182×10^{-3}	2.8693892×10^{-3}	4.900742×10^{-4}	8.488664×10^{-4}
I_3	0.659311367	0.6593440204	0.659343261	0.659326494	0.659328112
	2.811856×10^{-5}	2.140659×10^{-5}	2.025484×10^{-5}	5.175461×10^{-6}	2.72157×10^{-6}
I_4	0.85677642	0.8743924963	0.8739828201	0.866016272387	0.866785002
	1.17611×10^{-2}	8.557947×10^{-3}	8.0854108×10^{-3}	1.1035118×10^{-3}	2.16829×10^{-4}

Adaptive Quadrature

The adaptive algorithm was first introduced by Kuncir (Kuncir, 1962), to enhance the accuracy of any numerical quadrature rule depending on step-size parameter h. Adaptive quadrature routine allows us to rely on a low-order quadrature rule and, then improve the accuracy by implementing such a low-order rule on a finer mesh of the integration interval. The mechanism of the adaptive quadrature rule is to iteratively refine the step size of the relevant quadrature rule until a termination criterion is met and reaches the desirable degree of precision (Lyness, 1969). That is, the adaptive algorithm takes the following steps:

1. Set an allowed tolerance as $\varepsilon = 10^{-5}$and let $I_{Exact}(f) = E$.
2. At the mid-point $\rho = \frac{a+b}{2}$, subdivide the interval of integration $[a, b]$ into two subintervals $[a, \rho]$ and $[\rho, b]$.
3. Then implement the quadrature rule separately on each subinterval $[a, \rho]$ and $[\rho, b]$, to respectively obtain the approximate results R_1 and R_2.
4. Then estimate the error of the obtained approximate results as:
5. $|E - S_1|$ and $|E - S_2|$
6. If $|E - S_1| \le \varepsilon$, the termination criterion is met for the subinterval, the adaptive routine will stop, the same for another subinterval $[\rho, b]$.

If $|E - S_i| > \varepsilon, i = 1{,}2$, then the subdivision processes are still ongoing till the termination criterion is met.

We build up an adaptive algorithm by Mathematica 13.1 to produce the results shown in Tables (3). This table shows that the approximate values of the four integrals I_1, I_2, I_3, and I_4 computed respectively by the dual $I_{DH}(f)$ and its ingredients rules $I_{MI}(f)$ and $I_{aG3}(f)$ in an adaptive environment.

Table: (3). Numerical results computed by the hybrid quadrature rules $I_{HD}(f)$ compared with its constituent rules $I_{MI}(f)$ and $I_{aG3}(f)$ in adaptive environment.

Integral	$I_{MI}(f)$	steps	$I_{aG3}(f)$	Steps	$I_{HD}(f)$	steps
I_1	255.675599	5	255.7257015	3	255.67606135	5
I_2	0.13525742	3	0.135257133	3	0.135257140	3
I_3	0.65932983	2	0.65932996	2	0.6593299577	2
I_4	0.86697254	3	0.866973326	3	0.866973308	2

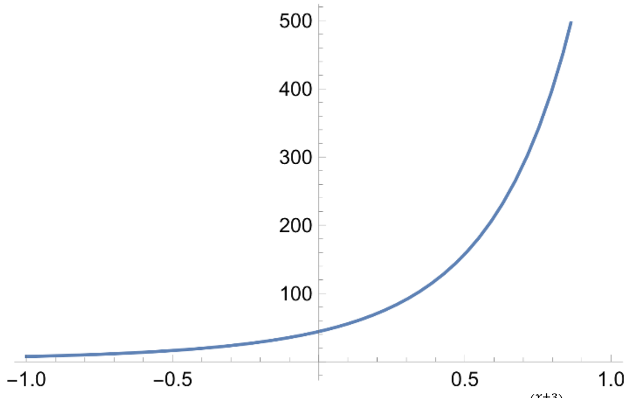

Figure (1): Plot of the integrand of $\int_{-1}^{1} \frac{e^{e^{\left(\frac{x+3}{2}\right)}}}{2} dx$

DISCUSSION

The error analysis of the proposed dual and triple hybrid rules analytically confirms that the degree of precision of such generated rules is higher than their ingredient quadrature rules as shown by equations (13) and (17). Also, the numerically observed results agree with the analytic error analysis. Moreover, the obtained numerical values for the four integrals I_1, I_2, I_3, and I_4 by implementing the dual hybrid rule $I_{DH}(f)$ are better than those attained by its ingredients $I_{MI}(f)$ and $I_{aG3}(f)$ as shown in Table (2). Also, the obtained numerical values of the four integrals I_1, I_2, I_3, and I_4 by using the triple hybrid rule $I_{TH}(f)$ are better than those obtained by its ingredients $I_{CDM}(f)$ and $I_{DH}(f)$ as shown in Table (2). Apart from the integral I_1, all the numerical values of I_2, I_3, and I_4 are reasonably very well although we use a quite few quadrature points. The slow convergence of the integral I_1 is due to the large variation of the integrand on the integration interval, because the integrand has sharp variation from the value 40 at $x = -1$ to the value 500 at the value $x = 1$ as shown in Figure 1. Thus, we easily tackle this issue by implementing the obtained quadrature rule in an adaptive environment with the allowed tolerance set to $\varepsilon = 10^{-5}$. Hence we achieve accurate results that coincide with the exact ones up to four digits only in two steps for the adaptive algorithm for the integrals I_3 and I_4 as shown in Table (3).

CONCLUSION

To conclude triple and dual hybrid quadrature rules have been constructed by blending Gauss-type rules with the classical or the modified Newton-Cotes-type rules that incorporate odd derivatives. Such a mixture incorporates the advantages of both types of quadrature rules to gain better accuracy, thus there is no need to increase the number of quadrature points that may bring instability issues to the numerical process. The acquired results have been enhanced by the adaptive quadrature algorithm. The analytic error analysis and the numerical computations both confirm the efficiency of the proposed approaches. A similar analogy can be adopted to generate hybrid quadrature rules of high-order accuracy by blending as many quadrature rules provided that they are of the same degree of precision.

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