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Fuzzy Rough Shortest Path problems

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Abstract:

In this paper, we are concerned with the design of a model and an algorithm for computing the shortest path in a network having triangular fuzzy number (triangular fuzzy rough number) arc lengths. First, α-cuts and ranks for each arc are used to find all possible path lengths. In a proposed algorithm, Euclidean distance is used to find the shortest path. Consequently, a shortest path is obtained from source node to destination node. Examples are worked out to illustrate the applicability of the proposed approach.

Keywords: Triangular fuzzy rough numbers, Shortest path, Fuzzy rough shortest distance.

1. Introduction:

The shortest path problem, which concentrates on obtaining the shortest path between a specified starting node and other destination nodes, is one of the most important and fundamental combinatorial network optimization problems in graph theory that have appeared in several real-world applications as a sub problem. However, in real life scenarios, many kinds of uncertainty are generally encountered, because of imperfect data or other reasons. In real world applications of shortest path problem, like scheduling, transportation, etc. Which are related to environmental scenarios and on-board weights could be highly uncertain due to fluctuating weather or traffic conditions. Therefore, finding the exact optimal path in such networks could be a challenge. In such cases, many experts use the fuzzy shortest path problem to manage the uncertainties of the shortest path problem due to randomness. Dubois and

Prade (1980) first analyzed this problem and proposed an algorithm to find the shortest path. They reported some solutions for the classical FSP problem by the use of extended sum and extended min and max operators. Klein (1991) presented a dynamic programming recursion algorithm in order to find the path(s) related to the threshold of a membership degree set by the decision maker. Chanas *et al.* (1994) proposed an approach based on the α-cut concept. In addition, Furukawa (1995) introduced an approach based on parametric orders. Okada and Spore (2000) presented an algorithm based on the order relation for a fuzzy network problem with L-R fuzzy numbers. They defined nondominated or Pareto optimal paths from the specified node to every other node. Chuang and Kung (2005) presented another algorithm to find the shortest path based on the idea of a minimum crisp number, if and only if any the other number is greater than or equal to this number, they developed this idea to the fuzzy shortest path length. Kung and Chuang (2005) proposed a new algorithm to solve the shortest path problem with discrete fuzzy arc lengths. They developed a fuzzy shortest path length procedure by a fuzzy minimum algorithm. Moazeni (2006) represented a lexicographic order relation among fuzzy numbers. By using multiple labeling and Dijkstra's shortest path algorithms, she proposed a new algorithm to find a set of non-dominated paths, which is related to the extension principle concept. Okada and Gen (1993) considered a shortest path problem with a new definition for order relation between intervals. Ji *et al.* (2007) proposed three concepts of fuzzy shortest path: expected shortest path, α-shortest path and the most shortest path, and formulate three models for the fuzzy shortest path according to difference decision criteria. Dey *et al.* (2018) have proposed an algorithmic approach based on genetic algorithm for finding shortest path from a source node to a destination node in a fuzzy graph with interval type-2 fuzzy arc lengths. Lin *et al.* (2021) presented a fuzzy mathematical model for FSP problem. They have used the CPLEX software to solve this FSP problem. They introduce an algorithmic method based on genetic algorithm for solving this FSP problem. This study will demonstrate that it is more effective to propose and use fuzzy rough shortest path to solve many real world problems where uncertain exist.

This paper is organized as follows: In Section 2 some basic definitions and some arithmetic results are presented. In Section 3, formulation of Shortest path with fuzzy arcs and application for solving FSP problem are established.

An algorithm solution for fuzzy rough shortest path problem is proposed in Section 4. Finally, the conclusion part is present in Section 5.

2. CONCEPTS

***Fuzzy Set*:** A fuzzy set A of a universal set X is defined by its membership function $\mu_A : X \rightarrow [0,1]$ which assigns a real number $\mu_A(x)$ in the interval [0, 1] to each element x ∈ X, where the value of $\mu_A(x)$ at x shows the grade of membership of x in A.

Triangular fuzzy number: A fuzzy number $\tilde{A}$ is said to be a triangular fuzzy numbers (T.F.N) if it has the following membership function:

$$\mu_A(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ \dfrac{x-a_3}{a_2-a_3} & a_2 \le x \le a_3 \\ 0 & otherwise \end{cases}$$

Then we say that $\tilde{A}$ is triangular fuzzy number, written as:

$$\tilde{A} = (a_1, a_2, a_3) \quad where \quad a_1, a_2, a_3 \in \mathcal{R}$$

A triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is called positive (negative) where $a_1 > 0 \ (a_3 < 0)$.

For $\alpha \in [0,1]$ the interval of confidence for the T.F.N $\tilde{A} = (a_1, a_2, a_3)$ at $\alpha - level\ set$ is defined as:

$$A_\alpha = [a_1^L(\alpha), a_3^U(\alpha)] = [a_1 + (a_2 - a_1)\alpha \,,\, a_3 + (a_2 - a_3)\alpha].$$

Euclidean Distance

Let $A = [x_1, y_1]$ and $B = [x_2, y_2]$ be two intervals. Then the Euclidean distance $D(A,B)$ is defined as $D\ (A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Minimum value of α- cuts

Let $A_\alpha = [a,b]$ and $B_\alpha = [p,q]$ be two α-cuts. The minimum value of A_α and B_α is given by $MV = min\,[A_\alpha, B_\alpha] = [min\,(a,p)\,, min(b,q)]$.

Definition A ranking function is a function $\mathcal{R}: F(R) \longrightarrow R$, where F(R) is a set of fuzzy numbers defined on set of real numbers, which maps each fuzzy number into the real line, where a natural order exists. Let $\tilde{A} = (a, b, c)$ be a triangular fuzzy number then

$$\mathcal{R}(\tilde{A}) = \frac{(a + 2b + c)}{4}$$

Fuzzy rough number A fuzzy rough number $\tilde{A}^R$ is a triangular fuzzy rough number denoted by $\tilde{A}^R = [(a^{LL}, a^M, a^{UL}) : (a^{LU}, a^M, a^{UU})]$ where $a^{LU}, a^{LL}, a^M, a^{UL}$ and $a^{UU} \in \mathcal{R}$ such that $a^{LU} \leq a^{LL} \leq a^M \leq a^{UL} \leq a^{UU}$ and the membership function can be defined as:

$$\mu_{\tilde{A}^R}(x) = \begin{cases} \mu_{\tilde{A}^L}(x) = \begin{cases} \dfrac{x - a^{LL}}{a^M - a^{LL}} & a^{LL} \leq x \leq a^M \\ \dfrac{a^{UL} - x}{a^{UL} - a^M} & a^M \leq x \leq a^{UL} \\ 0 & otherwise \end{cases} \\ \mu_{\tilde{A}^U}(x) = \begin{cases} \dfrac{x - a^{LU}}{a^M - a^{LU}} & a^{LU} \leq x \leq a^M \\ \dfrac{a^{UU} - x}{a^{UU} - a^M} & a^M \leq x \leq a^{UU} \\ 0 & otherwise \end{cases} \end{cases}$$

Note that $\tilde{A}^L = (a^{LL}, a^M, a^{UL})$, $\tilde{A}^U = (a^{LU}, a^M, a^{UU})$ and $\tilde{A}^L \widetilde{\subseteq} \tilde{A}^U$. Where $\mu_{\tilde{A}^L}(x)$ and $\mu_{\tilde{A}^U}(x)$ are membership functions of lower and upper approximation triangular fuzzy number respectively. The membership function of triangular fuzzy rough number is shown in figure 1.

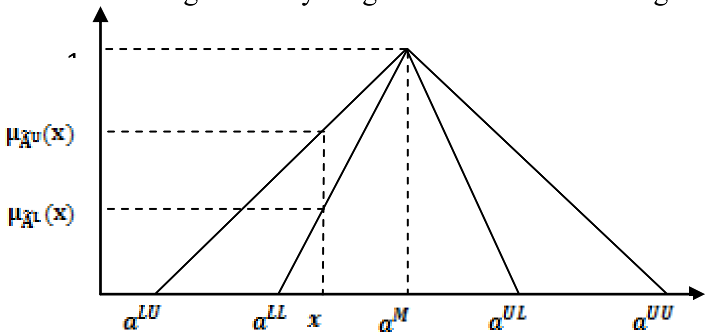

Figure 1: Membership function of the triangular fuzzy rough number.

Definition A ranking function is a function $\mathcal{R}: f(R) \longrightarrow [a, b]$, where $f(R)$ is a set of all fuzzy rough numbers defined on a set of real numbers, which maps each fuzzy rough number into the interval, where a natural order exists.

Let $\tilde{A}^R = [\tilde{A}^L : \tilde{A}^U] = [(a^{LL}, a^M, a^{UL}) : (a^{LU}, a^M, a^{UU})]$ be a triangular fuzzy rough number then

$$\mathcal{R}(\tilde{A}^R) = [\min\left(\mathcal{R}(\tilde{A}^L), \mathcal{R}(\tilde{A}^U)\right), \max\left(\mathcal{R}(\tilde{A}^L), \mathcal{R}(\tilde{A}^U)\right)]$$

3. Shortest path with fuzzy arcs

Consider a directed network G (V, E), consisting of a finite set of nodes V = 1,…, n and a set of m directed arcs E ⊆ V × V. Each arc is denoted by an ordered pair (*i*,*j*), where $i,j \in V$. Each arc is assigned to triangular fuzzy number. We calculate the α-cuts for each and every arc in the network by the formula $A_\alpha = [a_1 + (a_2 - a_1)\alpha,\ a_3 + (a_2 - a_3)\alpha]$ where $A = (a_1, a_2,\ a_3)$ is a triangular fuzzy number and α ∈ [0,1] .

Algorithm

Step: 1 Finding possible paths

(i) Find all the possible paths P_i from source node to destination node in the given acyclic network.

(ii) Assign the number of possible paths in the given acyclic network to N

Step: 2 Computation of length of paths

(i) Set α value between 0 and 1.

(ii) Find α -cuts for every arc.

(iii) Find the lengths L_i of all possible paths by adding the α-cuts of the corresponding arcs .

Step: 3 Comparison of paths

(i) Let $L_{min} = L_1$

(ii) For i = 2 to N MV = min (Lmin ,L_i)

D1 = D (MV, L_{min}) D2 = D (MV, L_i)

If D1 < D2 then $L_{min} = L_{min}$ Otherwise $L_{min} = L_i$

Step:4 Shortest Path

Shortest path is the corresponding path of L_{min}.

EXAMPLE 1

Consider the network in Fig.2 with 9 nodes and 12 arcs having various arc lengths (triangular fuzzy numbers), find the shortest path from source node 1 to destination node 9.

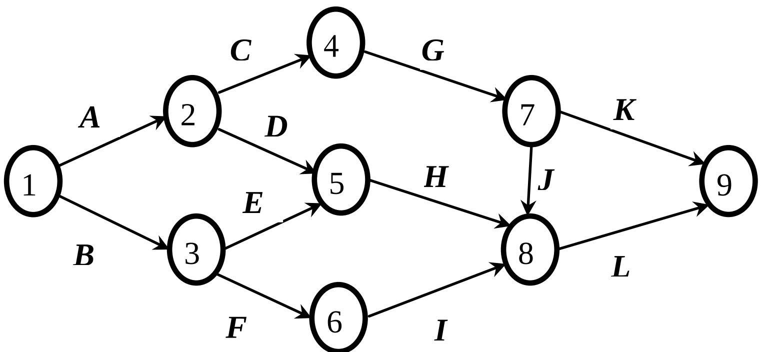

Fig. 2. Example Network

Table 1: The arc lengths

Arc	*Fuzzy number*	*Arc*	*Fuzzy number*
A	*(3,4,7)*	*G*	*(14,25,30)*
B	*(12,25,36)*	*H*	*(8,10,16)*
C	*(10,36,42)*	*I*	*(18,28,33)*
D	*(9,16,28)*	*J*	*(16,20,27)*
E	*(19,25,31)*	*K*	*(11,16,20)*
F	*(6,9,12)*	*L*	*(18,22,30)*

Step:1 Finding possible paths

(i) Find all possible paths P_i from the source node to destination node in the given acyclic network.

There are five possible paths

$$\begin{array}{ll} P_1: & A \rightarrow C \rightarrow G \rightarrow K \\ P_2: & A \rightarrow C \rightarrow G \rightarrow J \rightarrow L \\ P_3: & A \rightarrow D \rightarrow H \rightarrow L \\ P_4: & B \rightarrow E \rightarrow H \rightarrow L \\ P_5: & B \rightarrow F \rightarrow I \rightarrow L \end{array}$$

(ii) Assign the number of possible paths in the given acyclic network to N.

Here the number of possible paths are 5, then N = 5.

Step:2 Computation of length of paths

(i) Set α value between 0 and 1. Let $\alpha = 0.5$

(ii) Find α-cuts for every arc

Table 2: The *α-cuts* for every arc

Arc	*α-cuts*	*Arc*	*α-cuts*
A_α	*[3.5, 5.5]*	G_α	*[19.5, 27.5]*
B_α	*[18.5, 30.5]*	H_α	*[9, 13]*
C_α	*[23, 39]*	I_α	*[23, 30.5]*
D_α	*[12.5, 22]*	J_α	*[18, 23.5]*
E_α	*[22, 28]*	K_α	*[13.5, 18]*
F_α	*[7.5, 15.5]*	L_α	*[20, 26]*

ii) Find the lengths L_i of all possible paths by adding the α-cuts of the corresponding arcs

$$L_1 = [59.5, 90]$$
$$L_2 = [84, 121.5]$$
$$L_3 = [45, 66.5]$$
$$L_4 = [69.5, 97.5]$$
$$L_5 = [54, 102.5]$$

Step:3 Comparison of paths

(i) Let $L_{min} = L_1 \Rightarrow L_{min} = [59.5, 90]$

(ii) For i = 2 to 5 , $MV = \min[L_{min}, L_i]$

When i = 2 , $MV = \min[L_{min}, L_2] \Rightarrow MV = \min[[59.5, 90], [84, 121.5]] = [59.5, 90]$

$D_1 = D(MV, L_{min}) = D([59.5, 90],[59.5, 90])$

$$= \sqrt{(59.5 - 59.5)^2 + (90 - 90)^2} = 0$$

$D_2 = D(MV, L_2) = D([59.5, 90], [84, 121.5])$

$$= \sqrt{(59.5 - 84)^2 + (90 - 121.5)^2} = 39.9$$

$$If\ D_1 < D_2 \quad then\ L_{min} = L_{min}. \quad Otherwise\ L_{min} = L_i$$

$$Here\ D_1 < D_2. \qquad \therefore L_{min} = L_{min} = [59.5, 90]$$

When $i = 3$,

$MV = \min [L_{min}, L_3] = \min[[59.5, 90], [45, 66.5]] = [45, 66.5]$

$D_1 = D(MV, L_{min}) = D([45, 66.5], [59.5, 90])$

$$= \sqrt{(45 - 59.5)^2 + (66.5 - 90)^2} = 27.6$$

$D_2 = D(MV, L_3) = D([45, 66.5], [45, 66.5]) =$

$$\sqrt{(45 - 45)^2 + (66.5 - 66.5)^2} = 0$$

$D_1 > D_2$. Then $L_{min} = [45, 66.5] = L_3$

When $i = 4$

$MV = \min [L_{min}, L_4] = \min[[45, 66.5], [69.5, 97.5]] = [45, 66.5]$

$D_1 = D(MV, L_{min}) = D([45, 66.5], [45, 66.5]) = 0$

$D_2 = D(MV, L_4) = D([45, 66.5], [69.5, 97.5]) =$

$$= \sqrt{(45 - 69.5)^2 + (66.5 - 97.5)^2} = 39.5$$

$D_1 < D_2$. Then $L_{min} = [45, 66.5] = L_3$

When $i = 5$

$MV = \min [L_{min}, L_5] = \min[[45, 66.5], [54, 102.5]] = [45, 66.5]$

$D_1 = D(MV, L_{min}) = D([45, 66.5], [45, 66.5]) = 0$

$D_2 = D(MV, L_5) = D([45, 66.5], [54, 102.5])=$

$$\sqrt{(45 - 54)^2 + (66.5 - 102.5)^2} = 37.1$$

$D_1 < D_2$. Then $L_{min} = [45, 66.5] = L_3$

Step:4 Shortest Path

Shortest path is the corresponding path of L_{min}

$\therefore P_3$ is the shortest path. i.e., $P_3: \quad A \rightarrow D \rightarrow H \rightarrow L$ is the shortest path from source node to destination node.

4. Shortest path with fuzzy rough arcs

Now consider Each arc is assigned to triangular fuzzy rough number. We calculate the rank for each and every arc in the network by the formula as:

$$\mathcal{R}(\tilde{A}^{R}) = [\min\left(\mathcal{R}(\tilde{A}^{L}),\mathcal{R}(\tilde{A}^{U})\right),\max\left(\mathcal{R}(\tilde{A}^{L}),\mathcal{R}(\tilde{A}^{U})\right)]$$

where $\tilde{A}^{R} = [\tilde{A}^{L}:\tilde{A}^{U}] = [(a^{LL},a^{M},a^{UL}):(a^{LU},a^{M},a^{UU})]$ is a triangular fuzzy rough number.

EXAMPLE 2

Consider the network in Fig.3 with 9 nodes and 12 arcs having various arc lengths (triangular fuzzy rough numbers), find the shortest path from source node 1 to destination node 9.

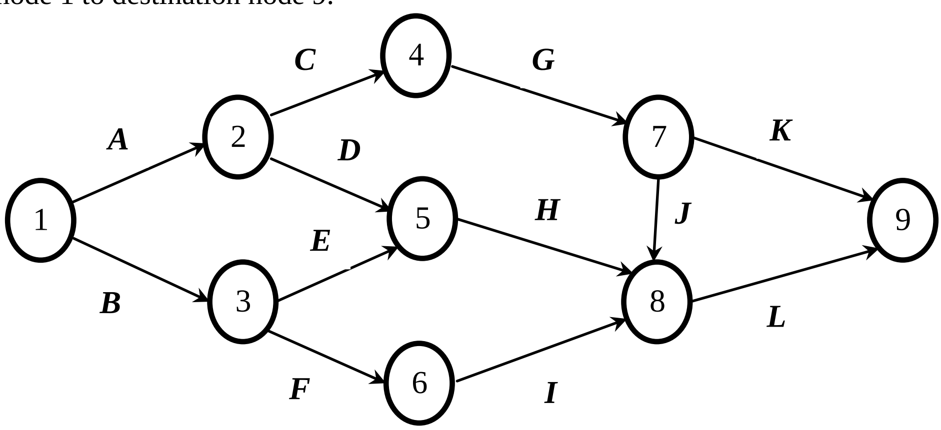

Fig. 3. Example Network.

Table 3: The arc lengths For Example 2

Arc	*Fuzzy rough number*	*Arc*	*Fuzzy rough number*
A	*[(3,4,7):(1,4,11)]*	*G*	*[(14,25,30):(9,25,38)]*
B	*[(12,25,36):(9,25,49)]*	*H*	*[(8,10,16):(5,10,23)]*
C	*[(10,36,42):(4,36,49)]*	*I*	*[(18,28,33):(12,28,42)]*
D	*[(9,16,28):(4,16,36)]*	*J*	*[(16,20,27):(12,20,35)]*
E	*[(19,25,31):(16,25,36)]*	*K*	*[(11,16,20):(9,16,26)]*
F	*[(6,9,12):(4,9,16)]*	*L*	*[(18,22,30):(16,22,40)]*

Step: 1 Finding possible paths

i) Find all possible paths P_i from the source node to destination node in the given acyclic network.

There are five possible paths

$$P_1: \quad A \rightarrow C \rightarrow G \rightarrow K$$
$$P_2: \quad A \rightarrow C \rightarrow G \rightarrow J \rightarrow L$$
$$P_3: \quad A \rightarrow D \rightarrow H \rightarrow L$$
$$P_4: \quad B \rightarrow E \rightarrow H \rightarrow L$$
$$P_5: \quad B \rightarrow F \rightarrow I \rightarrow L$$

ii) Assign the number of possible paths in the given acyclic network to N.

Here the number of possible paths are 5, then N = 5.

Step:2 Computation of length of paths

i) Find the rank of fuzzy rough number for every arc

Table 4: The rank for every arc

Arc	*Rank of Fuzzy rough number*	*Arc*	*Rank of Fuzzy rough number*
A	*[4.5, 5]*	*G*	*[23.5, 24.25]*
B	*[24.5, 27]*	*H*	*[11, 12]*
C	*[31, 31.25]*	*I*	*[26.75, 27.5]*
D	*[17.5, 18]*	*J*	*[20.75, 21.75]*
E	*[25, 25.5]*	*K*	*[15.75, 16.75]*
F	*[9, 9.5]*	*L*	*[23, 25]*

ii) Find the lengths Li of all possible paths by adding the ranks of the corresponding arcs

$$L_1 = [74.75, 77.25]$$
$$L_2 = [102.75, 107.25]$$
$$L_3 = [55.75, 60]$$
$$L_4 = [83.5, 89.5]$$
$$L_5 = [83.25, 89]$$

Step: 3 Comparison of paths

(i) Let $L_{min} = L_1 \Rightarrow L_{min} = [74.75, 77.25]$

(ii) For i = 2 to 5

$$MV = \min [L_{min}, L_i]$$

When i = 2

$MV = \min[L_{min}, L_2]$

$MV = \min[[74.75, 77.25], [102.75, 107.25]] = [74.75, 77.25]$

$D_1 = D(MV, L_{min}) = D([74.75, 77.25], [74.75, 77.25])$

$=\sqrt{(74.75 - 74.75)^2 + (77.25 - 77.25)^2} = 0$

$D_2 = D(MV, L_2) = D([74.75, 77.25], [102.75, 107.25])$

$= \sqrt{(74.75 - 102.75)^2 + (77.25 - 107.25)^2} = 41.03$

$If\ D_1 < D_2 \quad then\ L_{min} = L_{min}. \quad Otherwise\ L_{min} = L_i$

$Here\ D_1 < D_2. \qquad \therefore L_{min} = L_{min} = [74.75, 77.25]$

When $i = 3$,

$MV = \min [L_{min}, L_3] = \min[[74.75, 77.25], [55.75, 60]] = [55.75, 60]$

$D_1 = D(MV, L_{min}) = D([55.75, 60], [74.75, 77.25])$

$=\sqrt{(55.75 - 74.75)^2 + (60 - 77.25)^2} = 25.66$

$D_2 = D(MV, L_3) = D([55.75, 60], [55.75, 60]) = \sqrt{(55.75 - 55.75)^2 + (60 - 60)^2} = 0$ $D_1 > D_2$. Then

$L_{min} = [55.75, 60] = L_3$

When i = 4

$MV = \min [L_{min}, L_4] = \min[[55.75, 60], [83.5, 89.5]] = [55.75, 60]$

$D_1 = D(MV, L_{min}) = D([55.75, 60], [55.75, 60]) = 0$

$D_2 = D(MV, L_4) = D([55.75, 60], [83.5, 89.5]) = \sqrt{(55.75 - 83.5)^2 + (60 - 89.5)^2} = 40.5$

$D_1 < D_2$. Then $L_{min} = [55.75, 60] = L_3$

When i = 5

$MV = \min [L_{min}, L_5] = \min[[55.75, 60], [83.25, 89]] = [55.75, 60]$

$D_1 = D(MV, L_{min}) = D([55.75, 60], [55.75, 60]) = 0$

$D_2 = D(MV, L_5) = D([55.75, 60], [83.25, 89]) = \sqrt{(55.75 - 83.25)^2 + (60 - 89)^2} = 39.96$

$D_1 < D_2$. Then $L_{min} = [55.75, 60] = L_3$

Step: 4 Shortest Path

Shortest path is the corresponding path of L_{min}

∴ P_3 is the shortest path. i.e., $P_3: \quad A \rightarrow D \rightarrow H \rightarrow L$ is the shortest path from source node to destination node.

5. CONCLUSION

The shortest path problem is a very popular combinatorial network optimization problem in the field of graph theory. Many scientists work in several different kinds of shortest path problem. In the shortest path problem, a path between specified source s and destination t in a graph is determined by a decision maker such that the path length is minimum and traveling time within two nodes. In this paper, we presented an algorithm for computing the shortest path in an acyclic network using the α cuts, in which each edge is assigned to a triangular fuzzy number. In addition, we provided an algorithm for determining the shortest path in an acyclic network when each edge is assigned to a triangular fuzzy rough number. Consequently, the shortest path is obtained from source node to destination node.

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References

Chanas, S., Delgado, M., Verdegay, J. L., & Vila, M. A. (1995). Fuzzy optimal flow on imprecise structures. *European Journal of Operational Research*, *83*(3), 568-580.

Chuang, T. N., & Kung, J. Y. (2005). The fuzzy shortest path length and the corresponding shortest path in a network. *Computers & Operations Research*, *32*(6), 1409-1428.

Dey, A., Pradhan, R., Pal, A., & Pal, T. (2018). A genetic algorithm for solving fuzzy shortest path problems with interval type-2 fuzzy arc lengths. *Malaysian Journal of Computer Science*, *31*(4), 255-270.

Dubois, D., & Prade, H. (1980). Theory and applications, Academic Press, New York.

Furukawa, N. (1994). A parametric total order on fuzzy numbers and a fuzzy shortest route problem. *Optimization*, *30*(4), 367-377.

Ji, X., Iwamura, K., & Shao, Z. (2007). New models for shortest path problem with fuzzy arc lengths. *Applied Mathematical Modelling*, *31*(2), 259-269.

Klein, C. M. (1991). Fuzzy shortest paths. *Fuzzy sets and systems*, *39*(1), 27-41.

Kung, J. Y., & Chuang, T. N. (2005). The shortest path problem with discrete fuzzy arc lengths. *Computers & Mathematics with Applications*, *49*(2-3), 263-270.

Lin, L., Wu, C., & Ma, L. (2021). A genetic algorithm for the fuzzy shortest path problem in a fuzzy network. *Complex & Intelligent Systems*, *7*(1), 225-234.

Moazeni, S. (2006). Fuzzy shortest path problem with finite fuzzy quantities. *Applied Mathematics and Computation*, *183*(1), 160-169.

Okada, S., & Gen, M. (1993). Order relation between intervals and its application to shortest path problem. *Computers & industrial engineering*, *25*(1-4), 147-150.

Okada, S., & Soper, T. (2000). A shortest path problem on a network with fuzzy arc lengths. *Fuzzy sets and systems*, *109*(1), 129-140.