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The Iterative Bayawa Transform-Adomian Decomposition Method for Solving Various Classes of Linear and Nonlinear Integral Equations

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Abstract

Integral transforms can be directly applied to solve various classes of Volterra integral equations, including those with convolutional kernels and linear Volterra integro-differential equations. However, solving nonlinear Volterra and Fredholm integro-differential equations requires decomposing the integral transform with one of the established methods for solving integral equations, such as the Adomian Decomposition Method. In this paper, we demonstrate how the Bayawa integral transform, either independently or in combination with the Adomian decomposition method, can be employed to solve various classes of integral equations, depending on their type. To verify the method's efficiency, we solve diverse examples of integral equations, obtaining exact or approximate solutions based on their complexity. The obtained results show that the iterative Bayawa transform-Adomian decomposition approach effectively solves a wide range of IEs.

Keywords: Bayawa Transform, Adomian Decomposition Method, Volterra Integral Equations, Fredholm Integro-Differential Equations, Volterra Integro-Differential Equations, Combined Bayawa Transform-Adomian Decomposition.

INTRODUCTION

Integral transforms are powerful tools for solving various classes of mathematical problems involving differential and integral equations (IEs). Well-known transforms such as the Laplace, Fourier, Mellin, and Hankel transforms (Debnath & Bhatta, 2016) are widely used to solve linear and nonlinear ordinary and partial differential equations—including those of fractional order-as well as Volterra-type integral equations.

In addition to these classical transforms, several modern integral transforms have been introduced, including the Sumudu transform (Belgacem & Karaballi, 2006), the Anuj transform (Jafari et al., 2025), the ZZ transform (Sarah Th. Alaraji, 2025), and the El-Zaki transform (M. Al-Bugami et al., 2025), and recently the Bayawa transform (Zayyanu B. Bayawa & Aisha A. Haliru, 2024).

It is well established that integral transforms can directly solve linear and nonlinear Volterra IEs and linear Volterra Integro-Differential Equations (IDEs), provided they possess a difference (or convolution) kernel (Polyanin, 1998; Wazwaz, 2011, 2015). However, for other classes of IEs—such as linear and nonlinear Fredholm IEs, Fredholm IDEs, and nonlinear Volterra IDEs—the integral transform needs to be combined with any of the established analytical methods. These include the Adomian Decomposition Method (ADM) (Wazwaz, 2011), the Homotopy Perturbation Method (HPM) (Wazwaz, 2011), and the Variational Iteration Method (VIM). (Wazwaz, 2011).

Numerous hybrid approaches have been developed by integrating integral transforms with these techniques. Examples include the Sumudu homotopy method (Singh & Kumar, 2011), the Laplace VIM (Khuri & Sayfy, 2012; Liu et al., 2013), the Laplace decomposition method (Eshkuvatov, 2024; Wazwaz, 2010), the El-Zaki HPM (Alshehry et al., 2023), and the El-Zaki decomposition method (Chanchlani et al., 2023), all of which have been successfully applied to solve a wide range of differential and integral equations.

This paper demonstrates the applicability of the Bayawa transform in solving a wide range of linear and nonlinear IEs. Specifically, we address both first and second kinds of Fredholm and Volterra IEs, as well as Fredholm and Volterra integro-differential equations. Furthermore, we extend the method to solve linear and nonlinear generalized Abel's integral equations of the first and second kinds. To illustrate the method's versatility, we provide numerous examples that clarify when the Bayawa transform can be applied directly and when it must be combined with the ADM.

The structure of this paper is as follows: The related literature was reviewed in the introduction section. Then the definition of the Bayawa integral transform with some of its properties such as the linearity property, the Bayawa transform of derivatives, and the convolution theorem were introduced in section two. The third section; considers solving Volterra IEs of convolutional kernel by the Bayawa transform. Respectively, in section four and five, we show how to solve linear Volterra integro-differential of convolution kernel and the generalized Abel's IEs by the Bayawa Transform. Section six shows the formalism of the combined Bayawa transform-ADM for solving IEs and IDEs of Fredholm type, nonlinear Volterra IEs, as well as linear and nonlinear Volterra IDEs. To verify the acquired approach, numerous examples are shown in this section followed by a discussion and conclusion in section seven.

The Bayawa Integral Transform

Definition 1 (Zayyanu B. Bayawa & Aisha A. Haliru, 2024)**:** If the function $f(t)$ is a piecewise continuous of exponential order on the interval $K \geq t \geq 0$, then the Bayawa integral transform denoted by the operator $\mathfrak{B}\{f(t)\}$ is defined over the set:

$$A = \{f(t) \mid \exists\, M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_i}}, \qquad if\ t \in (-1)^i \times [0, \infty), i = 0,1\}$$

by the following formula:

$$\mathfrak{B}\{f(t)\} = F(v) = v^2 \int_0^\infty f(t)\, e^{-\frac{t}{v^2}} dt\,, \qquad t \geq 0, \qquad \tau_1 \leq v \leq \tau_2 \qquad (1)$$

where the constant M must be finite, and τ_1, τ_2 may be infinite, and v is a real parameter. The set A presents the conditions that guarantee the existence of the Bayawa transform $F(v)$, that is

$$|f(x)| \leq Me^{\frac{|t|}{\tau_i}}, t \to \infty, K, M > 0.$$

The Inverse Bayawa Transform

Since the Bayawa transform of the function $f(t)$ is $F(v)$, then the inverse Bayawa transform of $F(v)$ is $f(t)$, that is,

$$\mathfrak{B}^{-1}\{F(v)\} = \mathfrak{B}^{-1}\left\{v^2 \int_0^\infty f(t)\, e^{-\frac{t}{v^2}} dt\right\} = f(t), \qquad t \geq 0, \qquad \tau_1 \leq v \leq \tau_2,$$

where $\mathfrak{B}^{-1}$ denotes the operator of the inverse Bayawa transform.

The Bayawa Integral Transform of Some Functions

Here we show how to find the Bayawa integral transform of some elementary functions.

Example 1: Find the Bayawa transform of the function $f(t) = \sqrt{t}$.

Solution: Using the definition of the Bayawa transform given by the equation (1), one has

$$\mathfrak{B}\{\sqrt{t}\} = v^2 \int_0^\infty t^{\frac{1}{2}} \cdot e^{-\frac{t}{v^2}} dt.$$

Now based on the Euler definition of gamma function, given as

$$\int_0^\infty t^{\alpha-1} \cdot e^{-\beta t} dt = \frac{\Gamma(\alpha)}{\beta^\alpha}, \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

one can easily obtains, $\mathfrak{B}\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2} v^5$. This result can be generalized as,

$$\mathfrak{B}\left\{t^{\pm\frac{m}{2}}\right\} = \pm\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}\right) v^{\pm m+4}. \qquad (2)$$

Similarly, we can derive the following result,

$$\mathfrak{B}\left\{t^{\pm\frac{m}{3}}\right\} = \pm\left(\frac{m}{3}\right)\Gamma\left(\frac{m}{3}\right) v^{\pm\frac{2m}{3}+4}. \qquad (3)$$

These relations will be used later in solving the generalized Abel's IEs.

In a similar fashion, we can find the Bayawa transform for all the elementary functions that are tabulated in Table 1.

Table (1): The Bayawa transform of some elementary functions.

$f(t)$	$\mathfrak{B}\{f(t)\} = F(v)$
$t^n, n \in \mathbb{N} \cup \{0\}$	$n!\, v^{2n+4}, n \in \mathbb{N} \cup \{0\}$
$e^{\pm at}$	$\frac{v^4}{1 \mp av^2}$
$e^{at}t^n$	$\frac{n!\, v^{2n+4}}{(1 - av^2)^{n+1}}$
$\sinh^2(at)$	$\frac{2a^2v^8}{1 - 4a^2v^4}$
$\sin(at)$	$\frac{av^6}{1 + a^2v^4}$
$\cos(at)$	$\frac{v^4}{1 + a^2v^4}$
$\sinh(at)$	$\frac{av^6}{1 - a^2v^4}$
$\cosh(at)$	$\frac{v^4}{1 - a^2v^4}$

Some Properties of the Bayawa Integral Transform

Here in this section, we present some important properties of the Bayawa integral transform that we need later in this article. These properties are easy to prove in a similar manner to other integral transforms.

- **The Linearity Property:** For any constants a and b, and the functions $f(t), g(t) \in A$ defined above we have

$$\mathfrak{B}\{af(t)+bg(t)\}=a\mathfrak{B}\{f(t)\}+b\mathfrak{B}\{g(t)\}. \tag{4}$$

This property holds for the inverse Bayawa transform as well.

- **Multiplication by t :** For any piecewise continuous function $f(t)$ of exponential order

we have,

$$\mathfrak{B}\{tf(t)\}=\frac{v^3}{2}\left[\frac{d}{dv}-v^2\right]F(v). \tag{5}$$

- **The Bayawa Transform of Derivatives**

Theorem 1 (Zayyanu B. Bayawa & Aisha A. Haliru, 2024): If the function $f(t)$ and its derivatives are piecewise continuous functions of exponential order with some initial conditions, then we have

$$\mathfrak{B}\{f'(t)\}=\frac{1}{v^2}F(v)-v^2f(0), \tag{6}$$

$$\mathfrak{B}\{f''(t)\}=\frac{1}{v^4}F(v)-v^2f'(0)-f(0), \tag{7}$$

$$\mathfrak{B}\{f^{(n)}(t)\}=\frac{1}{v^{2n}}F(v)-\sum_{k=0}^{n-1}v^{-2n+2k+4}f^{(k)}(0). \tag{8}$$

- **Convolution of the Bayawa Transform**

The most important property of the Bayawa transform is listed in the following theorem.

Theorem 2: Let $F(v)$ and $F(v)$ are respectively the Bayawa transforms of the functions $f(t)$ and $g(t)$, and the Bayawa convolution product of these functions is presented as,

$$(f*g)(t)=(g*f)(t)=\int_0^t f(\tau)g(\tau-t)d\tau=\int_0^t g(\tau)f(\tau-t)d\tau\,.$$

Then the Bayawa transforms of this convolution product can be easily obtained as,

$$\mathfrak{B}\{(f*g)(t)\}=\frac{\mathfrak{B}\{f(t)\}.\mathfrak{B}\{g(t)\}}{v^2}. \tag{9}$$

Next, we show how to solve some IEs of Volterra type by using the Bayawa transform.

Solving Volterra IEs of Convolution Kernel by the Bayawa Transform

In this section, we demonstrate the application of the Bayawa transform to solve both linear and nonlinear Volterra IEs of the first and second kind, provided the kernel is of convolution type. Additionally, we extend the method to linear Volterra integro-differential equations of both kinds. A key requirement for employing any integral transform is the validity of its convolution theorem. Thus, we restrict our analysis to IEs with convolution -type kernels to ensure the transform's applicability.

Definition 2 (Polyanin, 1998)**:** An integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral signs and can be presented as:

$$\mu\, u(x)=f(x)+\lambda\int_{a(x)}^{b(x)}K(x,t)F(u(t))dt, \tag{10}$$

where the function $a(x)$ and $b(x)$ are the limits of integration may be both variables, constants, or mixed. The functions $f(x),\mu$ and $K(x,t)$ are given for $a(x)\le x,t\le b(x)$, and λ is a non-zero parameter, which may be real or complex, the function $k(x,t)$ is called the kernel of integral equation. Respectively if $\mu=0$ or $\mu=c\neq 0$, then equation (10) is called an integral equation of the first or second kinds. The integral equation (10) is called non-linear if the function $F(u(t))$ is non-linear functions of $u(t)$ such as $u^2, u^3, \ldots, \sin u, e^u, \ln(u+1), \ldots$, otherwise the integral equation (10) is linear. If $f(x) = 0$ then it is called homogeneous, otherwise it is called nonhomogeneous IE.

Example 2: Consider the second-kind and linear Volterra IE as,

$$\mathrm{u}(x) = \sin(x) + \cos(x) + 2\int_0^x \sin(x-t)\,\mathrm{u}(\mathrm{t})\,dt. \qquad (11)$$

Solution: This IE has a convolution kernel, thus taking the Bayawa transform of this IE and making use of the convolution product (9) with the aid of Table 1 we obtain,

$$U(v) = \frac{v^6}{1+v^4} + \frac{v^4}{1+v^4} + \frac{2}{v^2}\left[\frac{v^6}{1+v^4}U(v)\right],$$

where $\mathfrak{B}\{u(x)\} = U(v)$. Now carrying out some manipulations, leads to

$$u(x) = \mathfrak{B}^{-1}\{U(v)\} = \mathfrak{B}^{-1}\left(\frac{v^4}{1-v^2}\right) = e^x.$$

Example 3: Consider the first-kind and non-linear Volterra IE as,

$$\frac{1}{2} + \frac{1}{6}\cosh(2x) - \frac{2}{3}\cosh(x) = \int_0^x \sinh(x-t)\,u^2(t)\,dt. \qquad (12)$$

Solution: Firstly, we need to linearize this IE by assuming that $w(x) = u^2(x)$, then this IE is converted to the following linear Volterra IE as,

$$\frac{1}{2} + \frac{1}{6}\cosh(2x) - \frac{2}{3}\cosh(x) = \int_0^x \sinh(x-t)\,w(t)\,dt.$$

This kernel of this IE is of a convolution type, thus taking the Bayawa transform of it and making use of the convolution theorem 2 with the aid of Table 1 leads to,

$$\frac{v^4}{2} + \frac{1}{6}\frac{v^4}{(1-4v^4)} - \frac{2}{3}\frac{v^4}{(1-v^4)} = \frac{1}{v^2}\left[\frac{v^4}{(1-v^4)}W(v)\right],$$

where $\mathfrak{B}\{w(x)\} = W(v)$. By carrying out some manipulations, one obtains

$$w(x) = \mathfrak{B}^{-1}\{W(v)\} = \mathfrak{B}^{-1}\left(\frac{2v^8}{1-4v^4}\right) = \sinh^2(x).$$

Thus, the solution of the original non-linear VIE (12) is obtained as, $u(x) = \sinh(x)$.

Example 4: Consider the first-kind and non-linear Volterra IE as,

$$\frac{1}{2}\sin(x) - \frac{1}{2}x\cos(x) = \int_0^x \sinh(x-t)\sin(u(t))\,dt. \qquad (13)$$

Solution: Firstly, we need to linearize this IE by assuming that,

$$w(x) = \sin(u(x)), \qquad u(x) = \sin^{-1}(w(x)).$$

Then this IE is converted to the following linear Volterra IE as,

$$\frac{1}{2}\sin(x) - \frac{1}{2}x\cos(x) = \int_0^x \sinh(x-t)\,w(t)\,dt.$$

Now taking the Bayawa transform of this IE and making use of the convolution theorem 2 with the aid of Table 1 leads to,

$$\frac{1}{2}\frac{v^6}{(1+v^4)} - \frac{1}{2}\mathfrak{B}\{x\cos(x)\} = \frac{1}{v^2}\left[\frac{v^6}{(1+v^4)}W(v)\right], \qquad (14)$$

where $\mathfrak{B}\{w(x)\} = W(v)$. Now recall the property (5) to obtain

$$\mathfrak{B}\{x\cos(x)\} = \frac{v^3}{2}\left[\frac{d}{dv} - v^2\right]\left(\frac{v^4}{1+v^4}\right) = \frac{v^6(1-v^4)}{(1+v^4)^2}. \qquad (15)$$

Now substituting from equation (15) into equation (14), then carrying out some manipulations, leads to

$$w(x) = \mathfrak{B}^{-1}\{W(v)\} = \mathfrak{B}^{-1}\left(\frac{v^6}{1+v^4}\right) = \sin(x).$$

Thus, the solution of the original IE (13) is obtained as, $u(x) = \sin^{-1}(\sin x) = x$.

Solving Linear Volterra Integro-Differential Equations of Convolution Kernel by the Bayawa Transform

In this section we show how to solve the linear Volterra integro-differential equation of convolution kernel directly by using the Bayawa transform.

Definition 3: The linear Volterra integro-differential equation of convolution kernel is given by,

$$u^{(n)}(t) = f(t) + \lambda \int_a^x K(x-t)u(t)dt, \qquad (16)$$

with initial condition $u^{(k)}(0) = d_n; 0 \le k \le n-1, k \in \mathbb{N}$, where $u^{(n)}(\mathrm{x}) = \frac{d^n u}{dx^n}$ and d_n are constants. We note that the unknown function $u(x)$ in the integro-differential integral equation occurs twice as an ordinary derivative and under the integral sign.

Example 5: Consider the first-kind and linear Volterra IDE of the second order as,

$$\frac{1}{2}\sin(2x) = \int_0^x (x-t)u(t)\ dt + \frac{1}{4}\int_0^x (x-t-1)u''(t))\ dt, \qquad (17)$$

with the initial conditions $u(0) = 1,\ u'(0) = 0$.

Solution: Taking the Bayawa transform of this IDE and making use of the convolution theorem 2 and the relation (7) with the help of Table 1 leads to,

$$\frac{1}{v^2}[v^6 U(v)] + \frac{1}{4v^2}(v^6 - v^4)\left[\frac{1}{v^4}U(v) - u(0) - v^2 u'(0)\right],$$

where $\mathfrak{B}\{u(x)\} = U(v)$. Now plugging in the given initial conditions into this equation and then carrying out some manipulations, leads to

$$\mathfrak{B}^{-1}\{U(v)\} = \mathfrak{B}^{-1}\left(\frac{v^4}{1+4v^4}\right).$$

Thus, one obtains the solution of the IDE (17) as,

$u(x) = \cos(2x)$.

2. Solving Generalized Abel's IEs by the Bayawa Transform

In this section we show how to solve the linear generalized Abel's integral equations that has a weakly-singular kernel by the Bayawa transform.

Definition 4: The generalized Abel's integral equation is defined as,

$$u(x) = f(x) + \int_0^x \frac{F(u(t))}{|x-t|^\rho}dt, \quad 0 < \rho < 1. \qquad (18)$$

It should be noted that Abel's integral equation is a special case of the equation (18) when $\rho = \frac{1}{2}$. The function $F(u(t))$ is assumed to be smooth, that is it has continuous derivatives of all orders. The linearity of this IE relying on the linearity of the function $F(u(t))$.

Since the generalized Abel's integral equations are of convolution kernel, thus they can be solved directly by the Bayawa transform relying on the convolution property (9) as shown in the next example.

Example 6: Consider the first-kind and non-linear Abel's IE as,

$$\frac{4}{3}x^{\frac{3}{2}} = \int_0^x \frac{\ln(u(t))}{\sqrt{x-t}}\,dt. \qquad (19)$$

Solution: Firstly, we need to linearize this IE by assuming that,
$w(x) = \ln(u(x))$, then $u(x) = e^{w(x)}$.
Thus, this IE is converted to the following linear IE as,

$$\frac{4}{3}x^{\frac{3}{2}} = \int_0^x \frac{\mathrm{w(t)}}{\sqrt{x-t}}\,dt.$$

Now taking the Bayawa transform of this IE and making use of the convolution theorem 2 in addition to the relation (2) leads to,

$$\frac{4}{3}\frac{3\sqrt{\pi}}{4}v^7 = \frac{1}{v^2}\mathfrak{B}\left\{x^{-\frac{1}{2}}\right\}W(v) = \frac{\sqrt{\pi}v^3}{v^2}W(v).$$

Then carrying out some manipulations, leads to
$w(x) = \mathfrak{B}^{-1}\{W(v)\} = \mathfrak{B}^{-1}(v^6) = x.$
Thus, the solution of the original non-linear Abel's IE (19) is obtained as, $u(x) = e^x$.

Example 7: Consider the linear Abel's integral equation of the first-kind as,

$$\frac{8}{3}x^{\frac{3}{2}} + \frac{16}{5}x^{\frac{5}{2}} + \int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt. \qquad (20)$$

Solution: We start by taking the Bayawa transform of this IE and making use of the convolution theorem 2 as,

$$\frac{8}{3}\mathfrak{B}\left\{x^{\frac{3}{2}}\right\} + \frac{16}{5}\mathfrak{B}\left\{x^{\frac{5}{2}}\right\} = \frac{1}{v^2}\mathfrak{B}\left\{x^{-\frac{1}{2}}\right\}U(v).$$

Now recall the relation (2) for the Bayawa transform of the involved functions, to obtain

$$\frac{8}{3}.\frac{3\sqrt{\pi}}{4}v^7 + \frac{16}{5}.\frac{15\sqrt{\pi}}{8}v^9 = \frac{1}{v^2}\sqrt{\pi}v^3U(v).$$

Then carrying out some manipulations, leads to
$U(v) = 2v^6 + 6v^8.$
Thus, one has
$u(x) = \mathfrak{B}^{-1}\{U(v)\} = \mathfrak{B}^{-1}(2v^6 + 6v^8) = 2x + 3x^2.$

Example 8: Consider the generalized Abel's integral equation of the second-kind as,

$$u(x) = x - \frac{9}{10}x^{\frac{5}{3}} + \int_0^x \frac{u(t)}{(x-t)^{\frac{1}{3}}}\,dt. \qquad (21)$$

Solution: We start by taking the Bayawa transform of this IE and making use of the convolution theorem 2 leads to,

$$U(v) = v^6 - \frac{9}{10}.\mathfrak{B}\left\{x^{\frac{5}{3}}\right\} + \frac{1}{v^2}\mathfrak{B}\left\{x^{-\frac{1}{3}}\right\}U(v).$$

Now recall the relation (3) for the Bayawa transform of the involved functions, to obtain

$$U(v) = v^6 - \frac{27}{16}\Gamma\left(\frac{2}{3}\right)v^{\frac{22}{3}} + \frac{1}{v^2}\Gamma\left(\frac{2}{3}\right)v^{\frac{10}{3}}U(v).$$

Then carrying out some manipulations, leads to
$u(x) = \mathfrak{B}^{-1}\{U(v)\} = \mathfrak{B}^{-1}(v^6) = x.$

The Combined Bayawa Transform-Adomian Decomposition Method

So far, we show the applicability of the Bayawa transform independently for solving IE of Volterra type with convolutional kernel. However, in this section we show how to hybridize the Bayawa transform with the ADM for solving IE and IDEs of Fredholm type, nonlinear Volterra IEs, and linear or nonlinear Volterra IDEs.

Solving Non-Linear Volterra Integro-Differential Equations

Definition 5: The non-linear Volterra integro-differential equation of the nth-order is given by,

$$u^{(n)}(x) = f(x) + \lambda \int_a^x K(x,t)\big[L\big(u(t)\big) + N\big(u(t)\big)\big]dt, \qquad (16)$$

with initial condition $u^{(k)}(0) = d_n; 0 \leq k \leq n-1, k \in \mathbb{N}$, where $u^{(n)}(x) = \frac{d^n u}{dx^n}$ and d_n are constants. Respectively, $L\big(u(t)\big)$ and $N\big(u(t)\big)$ are linear and nonlinear operators acting on the function $u(t)$.

Definition 6 (Wazwaz, 2011)**:** The Adomian polynomials for the non-linear term $F\big(u(t)\big)$ are given by the following formula:

$$A_n(t) = \frac{1}{n!}\frac{d^n}{dx^n}\left[F\left(\sum_{i=0}^{n} \lambda^i u_i(t)\right)\right], n = 0,1,2,\ldots \qquad (17)$$

By using this relation, one can obtain the Adomian polynomials as,

$$A_0(t) = F(u_0(t)), \quad A_1(t) = u_1(t)F'(u_0), \quad A_2(t) = u_2(t)F'(u_0) + \frac{1}{2!}u_1(t)F''(u_0), \ldots$$

To explain the idea of the iterative approach namely the combined Bayawa transform-ADM, firstly we take the Bayawa transform of the IDE (16) to obtain

$$\mathfrak{B}\{u^{(n)}(x)\} = \mathfrak{B}\{f(x)\} + \lambda\mathfrak{B}\left\{\int_a^x K(x,t)\big[L\big(u(t)\big) + N\big(u(t)\big)\big]dt\right\},$$

Now we recall the relation (8) and carrying on some calculations to obtain

$$U(v) = \underbrace{v^{2n}\sum_{k=0}^{n-1} v^{-2n+2k+4}u^{(k)}(0) + v^{2n}F(v)}_{W(v)} + \underbrace{v^{2n}\mathfrak{B}\left\{\int_a^x K(x,t)\big[L\big(u(t)\big) + N\big(u(t)\big)\big]dt\right\}}_{H(v)}. \qquad (18)$$

A necessary condition for this to comply is that,

$$\lim_{v\to 0} v^{2n}\mathfrak{B}\left\{\int_a^x K(x,t)\big[L\big(u(t)\big) + N\big(u(t)\big)\big]dt\right\} = 0$$

Taking the inverse Bayawa transform of the resultant IE (18) leads to

$u(x) = \mathfrak{B}^{-1}\{W(v)\} + \mathfrak{B}^{-1}\{v^{2n}F(v)\} + \mathfrak{B}^{-1}\{H(v)\}$.

To be able to proceed further, we need to treat this IE by the iterative ADM, to do so we decompose the linear function $L\big(u(t)\big)$ as,

$$L\big(u(t)\big) = \sum_{n=0}^{\infty} u_n(t). \qquad (19)$$

Whereas we decompose the nonlinear function $N\big(u(t)\big)$ as,

$$N\big(u(t)\big) = \sum_{n=0}^{\infty} A_n(t), \qquad (20)$$

where $A_n(t)$ are the Adomain polynomials defined by the equation (17). Thus, substituting the decompositions (19) and (20) back into the IE (18) yields,

$$\sum_{n=0}^{\infty} u_n(t) = \mathfrak{B}^{-1}\{W(v)\} + \mathfrak{B}^{-1}\{v^{2n}F(v)\} + \mathfrak{B}^{-1}\left\{v^{2n}\mathfrak{B}\left\{\int_a^x K(x,t)\sum_{n=0}^{\infty}[u_n(t) + A_n(t)]\,dt\right\}\right\}.$$

Finally, the ADM provides the following recursive relation as,

$$u_k(t) = \mathfrak{B}^{-1}\left\{v^{2n}\mathfrak{B}\left\{\int_a^x K(x,t)[u_{k-1}(t) + A_{k-1}(t)]dt\right\}\right\}, k = 1,2,3, \dots$$

Where the initial component is taken as,

$$u_0(t) = \mathfrak{B}^{-1}\{W(v)\} + \mathfrak{B}^{-1}\{v^{2n}F(v)\}.$$

The next example illustrates this iterative approach.

Example 9: Consider the non-linear and second-order Volterra IDE of the first kind as,

$$\int_0^x (x-t)\,[u^2(t) + u''(t)]\,dt = -\frac{1}{4} - 3x + \frac{1}{4}x^2 + 3\,\sinh(x) + \frac{1}{4}\cosh^2 x. \qquad (21)$$

with the initial conditions $u(0) = 1,\ u'(0) = 0$.

Solution: We start by taking the Bayawa transform of this IE using the relation (7) with the convolution theorem 2 to obtain,

$$\mathfrak{B}\left\{\int_0^x (x-t)\,u^2(t)\,dt\right\} + \frac{v^6}{v^2}\left[\frac{U(v)}{v^4} - u(0) - v^2u'(0)\right] = \mathfrak{B}\left\{-\frac{1}{4} - 3x + \frac{1}{4}x^2 + 3\,\sinh(x) + \frac{1}{4}\cosh^2 x\right\}.$$

Plugging the given initial conditions into this equation, to obtain

$$U(v) = v^6 + v^4 + \mathfrak{B}\left\{-\frac{1}{4} - 3x + \frac{1}{4}x^2 + 3\,\sinh(x) + \frac{1}{4}\cosh^2 x\right\} - \mathfrak{B}\left\{\int_0^x (x-t)\,u^2(t)\,dt\right\}$$

Now by taking the inverse Bayawa transform of this IE as,

$$u(x) = \frac{3}{4} - 2x + \frac{1}{4}x^2 + 3\,\sinh(x) + \frac{1}{4}\cosh^2(x) - \mathfrak{B}^{-1}\mathfrak{B}\left\{\int_0^x (x-t)\,u^2(t)\,dt\right\}$$

Now we resort to the ADM to treat the nonlinearity issue in the resultant IE. To do so, let

$$u_0(x) = \frac{3}{4} - 2x + \frac{1}{4}x^2 + 3\,\sinh(x) + \frac{1}{4}\cosh^2 x,$$

and

$$u_k(x) = \lambda\int_0^x (x-t)A_{k-1}(t)\,dt, \qquad k = 1,2,3, \dots$$

where the functions $A_k(t)$ are the Adomian polynomials defined by equation (17). Thus,

$$u_1(x) = \int_0^x (x-t)\,A_0(t)\,dt = \int_0^x (x-t)\,u_0^2(t)\,dt.$$

So, one has

$$u_1(x) = \frac{1}{96}(2x^4 - 32x^3 + 42x^2 - 288x + 3\cosh(2x) + 288\sinh(x) - 3).$$

Thus, the approximate solution converges to the exact solution as,

$$u(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^{2k+1}}{(2k+1)!} = 1 + \sinh(x).$$

Solving Different Classes of Fredholm Integral Equations

It is established that the integral transforms can only be applied to IEs of Volterra type with convolutional kernel, though by decomposing the considered integral transform with some established methods such as the ADM, we can apply integral transforms to solve all types of Fredholm IEs as we demonstrate here. Next, we present examples of the linear and non-linear Fredholm IEs and IDEs as well.

Example 10: Consider the linear Fredholm integral equation of the second-kind as,

$$u(x) = e^x - x + x\int_0^1 t\,u(t)\ dt. \qquad (22)$$

Solution: We start by taking the Bayawa transform of this IE as, so

$$U(v) = \frac{v^4}{1-v^2} - v^6 + \mathfrak{B}\left\{x\int_0^1 t\,u(t)\ dt\right\}.$$

Now by taking the inverse Bayawa transform of this IE as,

$$u(x) = e^x - x + \mathfrak{B}^{-1}\left\{\mathfrak{B}\left\{x\int_0^1 t\,u(t)\ dt\right\}\right\}$$

Now we resort to the ADM to solve this IE. To do so, let $u_0(x) = e^x - x$ and the recurrence relation obtained as,

$$u_k(x) = x\int_0^1 t\,u_{k-1}(t)\ dt, \qquad k = 1,2,3,\ldots$$

Thus,

$$u_1(x) = x\int_0^1 t\,(e^t - t)\ dt = \frac{2}{3}x, u_2(x) = \frac{2}{3}x\int_0^1 t^2\ \ dt = \frac{2}{9}x, \qquad u_3(x) = \frac{2}{27}x$$

Thus, the solution is

$$u(x) = u_0 + u_1 + u_2 + \cdots = e^x - x + \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right)$$

Taking the sum of the geometric series yields the exact solution as,

$$u(x) = e^x - x + \frac{2}{3}x.\frac{1}{1-\frac{1}{3}} = e^x.$$

Example 11: Consider the first-kind and linear Volterra IDE of the first order as,

$$u'(x) = 36x^2 + \int_0^1 u(t)\ dt, \qquad (23)$$

with the initial condition $u(0) = 1$.

Solution: Taking the Bayawa transform of this IDE leads to,

$$\frac{1}{v^2}U(v) - v^2u(0) = 72v^8 + \mathfrak{B}\left\{\int_0^1 u(t)\ dt\right\},$$

Now plugging the given initial condition into this equation and then carrying out some manipulations, leads to

$$U(v) = v^4 + 72v^{10} + v^2\mathfrak{B}\left\{\int_0^1 u(t)\ dt\right\},$$

Then taking the inverse Bayawa transform of this IE as,

$$u(x) = 1 + 12x^3 + \mathfrak{B}^{-1}\left\{v^2\mathfrak{B}\left\{\int_0^1 u(t)\ dt\right\}\right\}.$$

Now we resort to the ADM to solve this IE. To do so, let $u_0(x) = 1 + 12x^3$ and we have the following iterative relation as,

$$u_k(x) = \mathfrak{B}^{-1}\left\{v^2\mathfrak{B}\left\{\int_0^1 u_{k-1}(t)\ dt\right\}\right\}, \qquad k = 1,2,3,\ldots$$

Thus,

$$u_1(x) = \mathfrak{B}^{-1}\left\{v^2\mathfrak{B}\left\{\int_0^1 (1 + 12x^3)\ dt\right\}\right\} = \mathfrak{B}^{-1}\{v^2\mathfrak{B}\{4\}\} = 4x.$$

And similarly, we obtain $u_2(x) = 2x,\ \ u_3(x) = x$. Thus, one has

$$u(x) = u_0 + u_1 + u_2 + \cdots = 1 + 12x^3 + 4x\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right).$$

Taking the sum of the geometric series, leads to the exact solution of the IDE (23) as,

$u(x) = 1 + 2x + 12x^3$.

Example 12 (Wazwaz, 2011)**:** Consider the non-linear Fredholm IE of the second-kind as,

$$u(x) = 1 + \lambda + \lambda\int_0^1 [u^2(t) - u(t)]\ dt. \qquad (24)$$

Solution: We start by taking the Bayawa transform of this IE as,

$$U(v) = (1+\lambda)v^4 + \lambda\,\mathfrak{B}\left\{\int_0^1 [u^2(t) - u(t)]\ dt\right\}.$$

Now by taking the inverse Bayawa transform of this IE as,

$$u(x) = 1 + \lambda + \lambda\,\mathfrak{B}^{-1}\left\{\mathfrak{B}\left\{\int_0^1 [u^2(t) - u(t)]\ dt\right\}\right\}$$

Now we resort to the ADM to decompose both the linear and nonlinear functions appear in this IE. To do so, let $u_0(x) = 1 + \lambda$ and the iterative relation is obtained as,

$$u_k(x) = \lambda\int_0^1 [A_{k-1}(t) - u_{k-1}(t)]\ dt, \qquad k = 1,2,3,\ldots$$

where we treated the nonlinear term in the IE (24) by the Adomian polynomials. Thus,

$$u_1(x) = \lambda\int_0^1 [A_0(t) - u_0(t)]\ dt = \lambda\int_0^1 [u_0^2(t) - u_0(t)]\ dt = \lambda^3 + \lambda^2,$$

$$u_2(x) = \lambda\int_0^1 [A_1(t) - u_1(t)]\ dt = \lambda\int_0^1 [2u_0u_1 - u_1(t)]\ dt = 2\lambda^5 + 3\lambda^4 + \lambda^3,$$

$$u_3(x) = \lambda\int_0^1 [A_2(t) - u_2(t)]\ dt = \lambda\int_0^1 [2u_0u_2 + u_1^2 - u_2]\ dt = 5\lambda^7 + 10\lambda^6 + 6\lambda^5 + \lambda^4,$$

Thus, the approximate solution of the IE (24) obtained as,

$u(x) = u_0 + u_1 + u_2 + \cdots = 1 + \lambda + \lambda^2 + 2\lambda^3 + 4\lambda^4 + 9\lambda^5 + \cdots$

Solving Non-Linear Fredholm Integro-Differential Equations

Definition 7: The non-linear Fredholm integro-differential equation of the nth-order is given by,

$$u^{(n)}(x) = f(x) + \lambda\int_a^b K(x,t)\big[L\big(u(t)\big) + N\big(u(t)\big)\big]dt, \qquad (25)$$

with initial condition $u^{(k)}(0) = d_n; 0 \le k \le n-1, k \in \mathbb{N}$, where $u^{(n)}(x) = \frac{d^n u}{dx^n}$ and d_n are constants. Respectively, $L\big(u(t)\big)$ and $N\big(u(t)\big)$ are linear and nonlinear functions of $u(t)$.

Example 13 (Eshkuvatov, 2024)**:** Consider the non-linear Fredholm IDE of the second-kind and first order as,

$$u'(x) = \cos(x) - \frac{\pi}{48}x + \frac{1}{24}\int_0^{\pi} xu^2(t)\ dt, \qquad u(0) = 0. \qquad (26)$$

Solution: We start by taking the Bayawa transform of this IE as,

$$\frac{1}{v^2}U(v) - v^2u(0) = \frac{v^4}{1+v^4} - \frac{\pi}{48}v^6 + \mathfrak{B}\left\{\frac{1}{24}\int_0^{\pi} x\, u^2(t)\ dt\right\}.$$

$$U(v) = \frac{v^6}{1+v^4} - \frac{\pi}{48}v^8 + v^2\,\mathfrak{B}\left\{\frac{1}{24}\int_0^{\pi} x\, u^2(t)\ dt\right\}.$$

Now by taking the inverse Bayawa transform of this IE, we obtain

$$u(x) = \sin(x) - \frac{\pi x^2}{96} + \mathfrak{B}^{-1}\left\{v^2\,\mathfrak{B}\left\{\frac{1}{24}\int_0^{\pi} x\, u^2(t)\ dt\right\}\right\}$$

Now we resort to the ADM to solve this IE. To do so, let $u_0(x) = \sin(x) - \frac{\pi x^2}{96}$ and

$$u_k(x) = \mathfrak{B}^{-1}\left\{v^2\,\mathfrak{B}\left\{\frac{x}{24}\int_0^{\pi} A_{k-1}(t)\ dt\right\}\right\}, \qquad k = 1,2,3,\ldots$$

where we treated the nonlinear term in the IE by the Adomian polynomials A_k. Thus,

$$u_1(x) = \mathfrak{B}^{-1}\left\{v^2\,\mathfrak{B}\left\{\frac{1}{24}\int_0^{\pi} x\, u_0^2(t)\ dt\right\}\right\} = \frac{x^2}{2}\left[\frac{7\pi}{288} - \frac{\pi^3}{1152} + \frac{\pi^7}{1105920}\right],$$

$$u_2(x) = \mathfrak{B}^{-1}\left\{v^2\,\mathfrak{B}\left\{\frac{x}{24}\int_0^{\pi} A_1(t)\ dt\right\}\right\} = \mathfrak{B}^{-1}\left\{v^2\,\mathfrak{B}\left\{\frac{x}{24}\int_0^{\pi} 2u_0(t)u_1(t)\ dt\right\}\right\},$$

Doing some calculations leads to,

$$u_2(x) = \frac{x^2}{192}\left[-\frac{7\pi}{18} + \frac{\pi^3}{9} - \frac{\pi^5}{288} - \frac{\pi^7}{4608} + \frac{\pi^9}{92160} - \frac{\pi^{13}}{6! \times 184320}\right]$$

Thus, the approximate solution of the IE (26) is obtained as,

$$u(x) = \sin(x) - \frac{\pi x^2}{96} + \frac{x^2}{2}\left[\frac{7\pi}{288} - \frac{\pi^3}{1152} + \frac{\pi^7}{1105920}\right] + \frac{x^2}{192}\left[-\frac{7\pi}{18} + \frac{\pi^3}{9} - \frac{\pi^5}{288} - \frac{\pi^7}{4608} + \frac{\pi^9}{92160} - \frac{\pi^{13}}{6!\,184320}\right] + \cdots$$

where the exact solution of the given IE is $u(x) = \sin(x)$. This solution is similar to the approximate solution obtained by the Laplace-decomposition method presented in the reference (Eshkuvatov, 2024).

DISCUSSION AND CONCLUSION

In this paper, we demonstrate that the Bayawa transform can be effectively applied to solve various types of IEs. The Bayawa transform can be utilized independently to solve Volterra IEs with convolutional kernels, whether they are linear or nonlinear, as illustrated by examples 1, 2, 3, 4, and 5. Also, the Bayawa transform can be implemented independently to solve either linear or non-linear generalized Abel's integral equations which are IEs of weakly-singular and convolutional kernels as illustrated by the Examples 6, 7, 8. However, for solving Volterra IEs with non-convolutional kernels, it is necessary to first reduce the integral equation to an equivalent initial value problem (IVP). The resulting IVP can then be directly solved using the Bayawa transform, as shown in the accompanying reference (Zayyanu B. Bayawa & Aisha A. Haliru, 2024). When it comes to nonlinear integro-differential equations, the Bayawa transform cannot solve these integral equations by itself.

To address this challenge, we combine the Bayawa transform with the ADM to create an iterative approach that effectively provides an exact solution for such integro-differential equations, as demonstrated in Example 9. Generally, integral transforms are primarily used to solve Volterra IEs rather than Fredholm IEs. However, in this paper, we illustrate how the Bayawa transform can be applied to tackle both linear and nonlinear IEs, as well as integro-differential equations of Fredholm type, in conjunction with the ADM. This is shown through the Examples 10, 11, 12, and 13. The iterative Bayawa-decomposition method is considered as a semi-analytic method for solving integro-differential equations, because in most cases gives an exact solution, however for some complex equations, we may only obtain an approximate solution. The Bayawa transform was proven as an efficient tool for solving the nonlinear and first-kind, Fredholm Volterra IE, even though such IE usually classified as an ill-conditioned problem since its solution is extremely sensitive to any change in the free term $f(x)$. This issue makes such IE difficult to solve, even if a solution exists, it may not be unique. However, this is issue is manageable if the kernel of the IE is singular, as shown by examples 6, 7 and 8 where we obtain a unique solution for a nonlinear and first-kind IE of weakly-singular kernel.

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