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Existence of Periodic Solutions for Neutral Nonlinear Dynamic Systems with Delay Using Shift Operators and Krasnoselskii's Fixed Point Theorem

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Abstract
In this research, we investigate the existence of periodic solutions for a class of neutral-type nonlinear dynamic systems with delay, described by the equation: $x^{\Delta}(t) = A(t)x(t) + \sum_{i=1}^{p} Q_i^{\Delta}(t, x(\delta_-(s,t))) + \int_{-\infty}^{t} \left(D(t,s) f(x(\delta_-(s,t)))\right) \Delta s$. To address this problem, we adopt a contemporary framework for periodicity based on shift operators, which extends traditional periodic concepts to a broader class of time scales. This modern shift-based perspective proves particularly advantageous for time scales where the additivity condition $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ and for a fixed $T > 0$ may not hold—a limitation that precludes the use of classical periodicity in non-uniform or non-additive time domains. Notably, this generalized notion of periodicity is well-suited for non-standard time scales such as the quantum time scale $\overline{q^{\mathbb{Z}}}$ and the Cantor-like union $\bigcup_{k=1}^{\infty} [3^{\pm k}, 2.3^{\pm k}] \cup \{0\}$, which do not admit a regular periodic structure in the conventional sense. To effectively examine the periodic behavior of such systems, particularly those involving q-difference dynamics, we construct a technical apparatus capable of analyzing periodic solutions under the shift-based setting. Central to this approach is the transformation of the differential system into an equivalent integral form, a step that necessitates consideration of the transition matrix associated with the homogeneous Floquet-type system: $y^{\Delta}(t) = A(t)y(t)$ This integral reformulation enables the application of Krasnoselskii's fixed point theorem, a foundational result in nonlinear operator theory, which facilitates the demonstration of fixed-point existence—and thereby confirms the presence of nontrivial periodic solutions within the system.

Keywords: Fixed point, Floquet theory, Krasnoselskii, periodicity, Shift operators, transition matrix.

INTRODUCTION

Since earlier times, distinguished recognition has been taken into account regarding the theory of neutral functional equations including delays. This is as a result of its vital promise of its applications in branches such as applied mathematics. There is very little scientific work done that deals with general time scales, but many studies focus on neutral differential equations on regular time scales, including discrete and continuous cases. A time scale is defined as a nonempty arbitrary closed subset of real numbers. The importance of the existence of periodic solutions is particularly relevant to biologists due to their application in population models. (Kaufmann & Raffoul, 2006) were among the first to define the concept of periodic time scales, requiring the additivity condition $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ and for a fixed $T > 0$. However, this condition excludes many important time scales of interest to biologists and scientists, which (Kaufmann & Raffoul, 2006) framework could not address. (Adivar, 2013) later introduced the concept of shift periodic operators to overcome these difficulties. Adivar's notion is extensively used in our work to establish the existence of periodic solutions. Further discussions on periodic solutions on regular time scales can be found in (Bodine, 2003; Bohner & Peterson, 2001; Raffoul, 2005). Additionally, studies by (Adıvar & Koyuncuoglu, 2013; Henriquez et al., 2012) explore the existence of periodic solutions for systems of delayed neutral functional equations using Sadovskii and Krasnoselskii's fixed point theorems. Time scales have also been applied to logistic equations modeling population growth. A detailed model construction is provided by (May, 1973). The equation:

$$x^{\Delta} = -a(t)x^{\sigma} + f(t)$$

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$$x^{\Delta} = [a(t) \ominus (f(t)x)]x.$$

is derived for the case $T = \mathbb{R}$, with analogous time-scale equations discussed by (DaCunha, 2004). Another application is a variant of the hematopoiesis model (Weng & Liang, 1995):

$$x^{\Delta}(t) = -a(t)x(t) + \alpha(t)\int_0^{\infty} B(s)e_{-\alpha}(t,s)\Delta s$$

where x(t) represents the number of red blood cells at time tt, and $t, \alpha, \beta, \gamma \in C(T, \mathbb{R})$are T-periodic, with B being a non-negative integrable function. This extends (Ważewska-Czyżewska & Lasota, 1976) red cell system on $\mathbb{R}$.

Throughout this paper, familiarity with time-scale calculus is assumed. For further reading, see (Bodine, 2003; Bohner & Peterson, 2001).

This investigation examines the following equation:

$$x^{\Delta}(t) = A(t)x(t) + \sum_{i=1}^{p} Q_i^{\Delta}\big(t, x(\delta_-(s,t))\big) + \int_{-\infty}^{t} \Big(D(t,s)f\big(x(\delta_-(s,t))\big)\Big)\Delta s$$

The methodology follows that of (Makhzoum et al., 2023), where the authors study the nonlinear neutral dynamic system:

$$x^{\Delta}(t) = A(t)x(t) + Q^{\Delta}\big(t, x(\delta_-(s,t))\big) + G\big(t, x(t), x(\delta_-(s,t))\big), t \in \mathbb{T}$$

by applying results from (Adıvar & Koyuncuoglu, 2013; DaCunha, 2004), the system is analyzed, and using Krasnoselskii's fixed point theorem, the existence of a nonzero periodic solution under suitable conditions is established.

Preliminaries

This section introduces basic definitions and properties of shift operators, drawn from (Adivar, 2013; Adivar & Raffoul, 2010). The notation $[a,b]_{\mathbb{T}}$ to indicate the set $[a,b] \cap \mathbb{T}$ shall be used. The intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$, and $(a,b)_{\mathbb{T}}$ are defined as such

Definition 1: Shift Operators

Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$, including a fixed number $t_0 \in \mathbb{T}^*$. We define shift operators δ_+that map from $[t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^*$ to $\mathbb{T}^*$ and satisfy the following properties:

1. Monotonicity: The functions $\delta_+$are strictly increasing with respect to their second arguments. If (T_0, t) and (T_0, u) are in the domain D_+, then $T_0 \leq t \leq u$ implies

$$\delta_+(T_0, t) \leq \delta_+(T_0, u).$$

2. Inverse Relationship: If (T_1,u) and (T_2,u) are in $D_-$with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$. Similarly, if (T_1,u) and (T_2,u) are in $D_+$with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$.
3. Identity Property: If $t \in [t_0,\infty)_{\mathbb{T}}$, then $(t,t_0) \in D_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in D_+$ and $\delta_+(t_0,t) = t$.
4. Inverse Operations: If $(s,t) \in D_+$, then $(s,\delta_+(s,t)) \in D_\mp$ and $\delta_\mp(s,\delta_+(s,t)) = t$.
5. Commutativity: If $(s,t) \in D_+$and $(u,\delta_+(s,t)) \in D_\mp$, then $(s,\delta_\mp(u,t)) \in D_+$ and

$$\delta_\mp(u,\delta_+(s,t)) = \delta_+(s,\delta_\mp(u,t)).$$

Given these properties, $\delta_+$is called the forward shift operator, and $\delta_-$is called the backward shift operator. Both operators are associated with the initial point t_0 on $\mathbb{T}^*$.

Example 1: Shift Operators on Various Time Scales

The following table shows the shift operators $\delta_+(s,t)$ on some time scales:

Time Scale $\mathbb{T}$	Initial Point t_0	Subset $\mathbb{T}^*$	$\delta_-(s,t)$	$\delta_+(s,t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t-s$	$t+s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t-s$	$t+s$
$q^{\mathbb{Z}} \cup \{0\}$	1	$q^{\mathbb{Z}}$	t/s	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$\sqrt{t^2-s^2}$	$\sqrt{t^2+s^2}$

Lemma 1: Properties of Shift Operators

Let δ_+be the shift operators associated with the initial point t_0. Then we have the following properties:

1. $\delta_-(t,t) = t_0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.
2. $\delta_-(t_0,t) = t$ for all $t \in \mathbb{T}^*$.
3. If $(s,t) \in D_+$, then $\delta_+(s,t) = u$ implies $\delta_-(s,u) = t$. Conversely, if $(s,u) \in D_-$, then $\delta_-(s,u) = t$ implies $\delta_+(s,t) = u$.
4. $\delta_+\big(t,\delta_-(s,t_0)\big) = \delta_-(s,t)$ for all $(s,t) \in D_+$with $t \ge t_0$.
5. $\delta_+(u,t) = \delta_+(t,u)$ for all $(u,t) \in ([t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}) \cap D_+$.
6. $\delta_+(s,t) \in [t_0,\infty)_{\mathbb{T}}$ for all $(s,t) \in D_+$with $t \ge t_0$.
7. $\delta_-(s,t) \in [t_0,\infty)_{\mathbb{T}}$ for all $(s,t) \in ([t_0,\infty)_{\mathbb{T}} \times [s,\infty)_{\mathbb{T}}) \cap D_-$.
8. If $\delta_+(s,\cdot)$ is Δ-differentiable in its second variable, then $\delta_+^{\Delta_t}(s,\cdot) > 0$.
9. $\delta_+(\delta_-(u,s),\delta_-(s,v)) = \delta_-(u,v)$ for all $(s,v) \in ([t_0,\infty)_{\mathbb{T}} \times [s,\infty)_{\mathbb{T}}) \cap D_-$ and

$$(u,s) \in ([t_0,\infty)_{\mathbb{T}} \times [u,\infty)_{\mathbb{T}}) \cap D_-.$$

10. If $(s,t) \in D_-$and $\delta_-(s,t) = t_0$, then $s = t$.

Definition 2: Periodicity in Shifts

Let $\mathbb{T}$ be a time scale with shift operators δ_+associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts δ_+if there exists a $p \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t) \in D_\mp$ for all $t \in \mathbb{T}^*$. The smallest such p is called the period of $\mathbb{T}$.

Example 2: Periodic Time Scales

The following time scales are not additive periodic but are periodic in shifts δ_+:

1. $\mathbb{T}_1 = \{\pm n^2 : n \in \mathbb{Z}\}$, with $\delta_{\pm}(P,t) = \begin{cases} (\sqrt{t} \pm \sqrt{P})^2 & \text{if } t > 0 \\ \pm P & \text{if } t = 0, P = 1, t_0 = 0. \\ -(\sqrt{-t} \pm \sqrt{P})^2 & \text{if } t < 0 \end{cases}$
2. $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, with $\delta_{+}(P,t) = P^{\pm 1}t, P = q, t_0 = 1$.
3. $\mathbb{T}_3 = \bigcup_{n\in\mathbb{Z}} [2^{2n}, 2^{2n+1}]$, with $\delta_{+}(P,t) = P^{\pm 1}t, P = 4, t_0 = 1$.
4. $\mathbb{T}_4 = \left\{\frac{q^n}{1+q^n} : q > 1, n \in \mathbb{Z}\right\} \cup \{0,1\}$, with $\delta_{\pm}(P,t) = \dfrac{q^{\left(\frac{\ln(t/(1-t)) + \ln(P/(1-P))}{\ln q}\right)}}{1+q^{\left(\frac{\ln(t/(1-t)) + \ln(P/(1-P))}{\ln q}\right)}}$,

$P = \dfrac{q}{1+q}$.

Corollary 1: Periodic Shifts

Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_+$with period P. Then we have:

$$\delta_+(P,\sigma(t)) = \sigma(\delta_+(P,t)) \quad \text{for all} \quad t \in \mathbb{T}^*$$

Example 3: Non-Periodic Time Scale

The time scale $\tilde{\mathbb{T}} = (-\infty, 0] \cup [1, \infty)$ cannot be periodic in shifts δ_+. If there were a $p \in (t_0, \infty)_{\tilde{\mathbb{T}}^*}$ such that $\delta_+(p,t) \in \tilde{\mathbb{T}}^*$, then the point $\delta_-(p,0)$ would be right scattered. However, $\delta_-(p,0) < 0$, which leads to a contradiction since every point less than 0 is right dense.

Definition 3: Periodic Function in Shifts

Let $\mathbb{T}$ be a P-periodic time scale in shifts. A real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts δ_+if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that:

$$(T,t) \in D_{\pm} \text{ and } f\left(\delta_{\pm}^{T}(t)\right) = f(t) \quad \text{for all} \quad t \in \mathbb{T}^*$$

where $\delta_{+}^{T}(t) = \delta_+(T,t)$. The smallest such T is called the period of f.

Example 4: Periodic Function

Let $\mathbb{T} = \mathbb{R}$ with initial point $t_0 = 1$. The function:

$$f(t) = \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi\right), t \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$$

is 4-periodic in shifts δ_+since:

$$f(\delta_{\pm}(4,t)) = \begin{cases} f(t \cdot 4^{\pm 1}) & \text{if } t \geq 0 \\ f(t/4^{\pm 1}) & \text{if } t < 0 \end{cases} = \sin\left(\frac{\ln|t| \pm 2\ln(1/2)}{\ln(1/2)}\pi\right) = \sin\left(\frac{\ln|t|}{\ln(1/2)}\pi \pm 2\pi\right)$$
$$= f(t)$$

Definition 4: Δ-Periodic Function in Shifts

Let $\mathbb{T}$ be a P-periodic time scale in shifts. A real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that:

1. $(T,t) \in D_+$for all $t \in \mathbb{T}^*$.
2. The shifts δ_{+}^{T} are Δ-differentiable with rd-continuous derivatives.
3. $f\left(\delta_{\pm}^{T}(t)\right)\delta_{+}^{\Delta T}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_{+}^{T}(t) = \delta_+(T,t)$.

The smallest such T is called the period of f.

Example 5: Δ-Periodic Function

The function $f(t) = \frac{1}{t}$ is Δ-periodic on $q^{\mathbb{Z}}$ with period $T = q$.

Theorem 1: Integral of Δ-Periodic Function
Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_+$with period $P \in (t_0, \infty)_{\mathbb{T}^*}$, and let f be a Δ-periodic function in shifts $\delta_+$with period $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose $f \in C_{rd}(\mathbb{T})$. Then:

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_+^T(t_0)}^{\delta_+^T(t)} f(s)\Delta s.$$

Unified Floquet Theory with Respect to New Periodicity Concept
In this section, we list some results from [2] for further use.

Homogeneous Case
Consider the regressive time-varying linear dynamic initial value problem:

$$x^\Delta(t) = A(t)x(t), x(t_0) = x_0$$

where $A: \mathbb{T}^* \to \mathbb{R}^{n\times n}$ is Δ-periodic in shifts with period T. If the time scale is additive periodic, then $\delta_+^\Delta(T, t) = 1$, and Δ-periodicity in shifts becomes the same as periodicity in shifts.
The solution of the system can be indicated by the equality:

$$x(t) = \Phi_A(t, t_0)x_0$$

where $\Phi_A(t, t_0)$, called the transition matrix for the system, is given by:

$$\Phi_A(t, t_0) = I + \int_{t_0}^{t} A(\tau_1)\Delta\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2)\Delta\tau_2\Delta\tau_1 + \cdots$$

The matrix exponential $e_A(t, t_0)$ is not always identical to $\Phi_A(t, t_0)$ due to:

$$A(t)e_A(t, t_0) = e_A(t, t_0)A(t)$$

being true in any case. However, the equality:

$$A(t)\Phi_A(t, t_0) = \Phi_A(t, t_0)A(t)$$

is seldom not. It is clear from the above that the condition $e_A(t, t_0) \equiv \Phi_A(t, t_0)$ holds only if the matrix A satisfies:

$$A(t) \int_{s}^{t} A(\tau)\Delta\tau = \int_{s}^{t} A(\tau)\Delta\tau A(t)$$

for the following result, the set:

$$P(t_0) = \left\{\delta_+^{(k)}(T, t_0), k = 0,1,2, \dots\right\}$$

is defined, and the function:

$$\Theta(t) = \sum_{j=1}^{m(t)} \delta_-\left(\delta_+^{(j-1)}(T, t_0), \delta_+^{(j)}(T, t_0)\right) + G(t)$$

where:

$$m(t) = \min\left\{k \in \mathbb{N}: \delta_+^{(k)}(T, t_0) \geq t\right\}$$

and:

$$G(t) = \begin{cases} 0 & \text{if } t \in P(t_0) \\ -\delta_-\left(t, \delta_+^{(m(t))}(T, t_0)\right) & \text{if } t \notin P(t_0) \end{cases}$$

For an additive periodic time scale, we always have $\Theta(t) = t - t_0$.

Theorem 2: Matrix Exponential Equation Solution
For a nonsingular, $n \times n$ constant matrix M, a solution $R: \mathbb{T} \to \mathbb{C}^{n\times n}$ of the matrix exponential equation:

$$e_R(\delta_+^T(t_0), t_0) = M$$

can be given by:

$$R(t) = \lim_{s \to t} \left(\frac{M^{1/T[\Theta(\sigma(t)) - \Theta(s)]} - I}{\sigma(t) - s} \right),$$

where I is the $n \times n$ identity matrix and Θ is as defined above.

Lemma 2: Unique Solution of Dynamic Matrix Initial Value Problem

Let $\mathbb{T}$ be a time scale and $P \in \mathcal{R}(\mathbb{T}^*, \mathbb{R}^{n \times n})$ be a Δ-periodic matrix-valued function in shifts with period T. Then the solution of the dynamic matrix initial value problem:

$$Y^\Delta(t) = P(t)Y(t), Y(t_0) = Y_0$$

is unique up to a period T in shifts. That is:

$$\Phi_P(t, t_0) = \Phi_P\left(\delta_+^T(t), \delta_+^T(t_0)\right)$$

for all $t \in \mathbb{T}^*$.

Corollary 2: Periodic Matrix Exponential

Let $\mathbb{T}$ be a time scale and $P \in \mathcal{R}(\mathbb{T}^*, \mathbb{R}^{n \times n})$ be a Δ-periodic matrix-valued function in shifts. Then:

$$e_P(t, t_0) = e_P\left(\delta_+^T(t), \delta_+^T(t_0)\right)$$

Theorem 3: Floquet Decomposition

Let A be a matrix-valued function that is Δ-periodic in shifts with period T. The transition matrix for A can be given in the form:

$$\Phi_A(t, \tau) = L(t)e_R(t, \tau)L^{-1}(\tau), \text{ for all } t, \tau \in \mathbb{T}^*$$

where $R: \mathbb{T} \to \mathbb{C}^{n \times n}$ and $L(t) \in C_{rd}^1(\mathbb{T}^*, \mathbb{R}^{n \times n})$ are both periodic in shifts with period T and invertible.

Theorem 4: Periodic Solution Existence

There exists an initial state $x(t_0) = x_0 \neq 0$ such that the solution of the homogeneous system is T-periodic in shifts if and only if one of the eigenvalues of the matrix:

$$e_R(\delta_+^T(t_0), t_0) = \Phi_A(\delta_+^T(t_0), t_0)$$

Nonhomogeneous Case

Consider the nonhomogeneous regressive nonautonomous linear dynamic initial value problem:

$$x^\Delta(t) = A(t)x(t) + F(t), x(t_0) = x_0$$

where $A: \mathbb{T}^* \to \mathbb{R}^{n \times n}$ and $F \in C_{rd}(\mathbb{T}^*, \mathbb{R}^n) \cap \mathcal{R}(\mathbb{T}^*, \mathbb{R}^n)$. We suppose both A and F are Δ-periodic in shifts with period T.

Theorem 5: Periodic Solution of Nonhomogeneous System

For any initial point $t_0 \in \mathbb{T}^*$ and for any function F that is Δ-periodic in shifts with period T, there exists an initial state $x(t_0) = x_0$ such that the solution of the nonhomogeneous system is T-periodic in shifts if and only if there does not exist a nonzero $z(t_0) = z_0$ and $t_0 \in \mathbb{T}^*$ such that the T-periodic homogeneous initial value problem:

$$z^\Delta(t) = A(t)z(t), z(t_0) = z_0$$

has a solution that is T-periodic in shifts.

For more details about Floquet theory based on the new periodicity concept on time scales, we refer readers to (Adıvar & Koyuncuoglu, 2013).

Main Result

Consider a time scale $\mathbb{T}$ that is periodic under shift operators with period $T > 0$. Denote by P_T the space of all n-dimensional vector functions $\mathbf{x}(t)$ defined on $\mathbb{T}$, which are periodic under shifts with period T. This space, equipped with the norm

$$\|\mathbf{x}\| = \max_{t\in[t_0,\delta_T^+(t_0)]_{\mathbb{T}}} |\mathbf{x}(t)|,$$

forms a Banach space. Here, $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.

For an $n \times n$ matrix-valued function $\mathbf{A}(t) = [\alpha_{ij}(t)]$, we define its norm as follows:

$$\|\mathbf{A}\| = \sup_{t\in[t_0,\infty)_{\mathbb{T}}} |\mathbf{A}(t)|,$$

where the matrix norm $|\mathbf{A}(t)|$ is given by

$$|\mathbf{A}(t)| = \max_{1\le i\le n} \sum_{j=1}^{n} |\alpha_{ij}(t)|$$

Now, consider the delay dynamic system defined on $\mathbb{T}$:

$$\mathbf{x}^\Delta(t) = \mathbf{A}(t)\mathbf{x}(t) + \sum_{i=1}^{p} Q_i^\Delta\big(t,\mathbf{x}(\delta^-(s,t))\big) + \int_{-\infty}^{t} D(t,s) f\big(\mathbf{x}(\delta^-(s,t))\big)\Delta s. \qquad (1)$$

We assume that the coefficient matrix $\mathbf{A}$ belongs to the class of right-dense continuous functions $C_{rd}(\mathbb{T}^*,\mathbb{R}^{n\times n})$, and the mappings Q_i, D, and f are also right-dense continuous.

We impose the following structural conditions:

(a) Matrix Function Periodicity:

$$\mathbf{A}(\delta_T^+(t))\cdot(\delta_T^+)^\Delta(t) = \mathbf{A}(t), \forall t \in \mathbb{T}^*.$$

(b) Functional Invariance of

$$Q{:}\, Q\left(\delta_T^+(t),\mathbf{x}\big(\delta^-(s,\delta_T^+(t))\big)\right) = Q\big(t,\mathbf{x}(\delta^-(s,t))\big).$$

(c) Integral Kernel Compatibility:

$$G\left(\delta_T^+(t),\mathbf{x}(\delta_T^+(t)),\mathbf{x}\big(\delta^-(s,\delta_T^+(t))\big)\right)\cdot(\delta_T^+)^\Delta(t) = G\big(t,\mathbf{x}(t),\mathbf{x}(\delta^-(s,t))\big).$$

We also assume that there exists some $t \in \mathbb{T}^*$ such that

$$Q^\Delta(t,0) + G(t,0,0) \neq 0. \qquad (2)$$

Moreover, the associated homogeneous system

$$\mathbf{z}^\Delta(t) = \mathbf{A}(t)\mathbf{z}(t) \qquad (3)$$

is non-critical, meaning that the only periodic solution in shifts $\delta_T^+$is the trivial one.

Lemma 3

Assume that conditions (a)-(b) and inequality (2) are satisfied. Then, a function $\mathbf{x}(t) \in P_T$ is a solution of system (1) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ if and only if it satisfies the integral representation:

$$\begin{aligned} x(t) \quad &= \sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big) + \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1} \\ &\times \left[\int_t^{\delta_+^T(t)} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u,x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s \quad \right]\Delta u\right], \end{aligned} \qquad (4)$$

Proof

Let x(t)∈ P_T denote a solution of (1) such that $\mathbf{x}(t_0) = \mathbf{x}_0$, and let $\Phi_A(t,t_0)$ represent the state transition matrix associated with system (3). The necessity component of the proof follows directly from established properties of such systems. To address the sufficiency condition, we proceed by utilizing equation (1), which allows us to derive the required relationship

$$\left[x(t)-\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)\right]^{\Delta}$$
$$=A(t)\left(x(t)-\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)\right)+A(t)\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)$$
$$+\int_{-\infty}^{t}(D(t,s)$$

since $\Phi_A(t,t_0)\Phi_A^{-1}(t,t_0)=I$, we have

$$0=\left(\Phi_A(t,t_0)\Phi_A^{-1}(t,t_0)\right)^{\Delta}=\Phi_A^{\Delta}(t,t_0)\Phi_A^{-1}(t,t_0)+\Phi_A(\sigma(t),t_0)\left(\Phi_A^{-1}(t,t_0)\right)^{\Delta}=$$
$$\left(A(t)\Phi_A(t,t_0)\right)\Phi_A^{-1}(t,t_0)+\Phi_A(\sigma(t$$

that is,

$$\left(\Phi_A^{-1}(t,t_0)\right)^{\Delta}=-\Phi_A^{-1}(\sigma(t),t_0)A(t). \quad (5)$$

If $x(t)$ is a solution to (1) with the condition $x(t_0)=x_0$, then

$$\left\{\Phi_A^{-1}(t,t_0)\left(x(t)-\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)\right)\right\}^{\Delta}=\left(\Phi_A^{-1}(t,t_0)\right)^{\Delta}\left(x(t)-\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)\right)$$
$$+\Phi_A^{-1}(\sigma(t),t_0)\left(x(t)-\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)\right)^{\Delta}$$
$$=-\Phi_A^{-1}(\sigma(t),t_0)A(t)\left(x(t)-\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)\right)$$
$$+\Phi_A^{-1}(\sigma(t),t_0)\Bigg[A(t)\left(x(t)-\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)\right)$$
$$+A(t)\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)++\int_{-\infty}^{t}\Big(D(t,s)f\big(x(\delta_-(s,t))\big)\Big)\Delta s\Bigg]$$
$$=\Phi_A^{-1}(\sigma(t),t_0)\left[A(t)\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)++\int_{-\infty}^{t}\Big(D(t,s)f\big(x(\delta_-(s,t))\big)\Big)\Delta s\right].$$

By integrating the last equality from t_0 to t, we arrive at

$$x(t)=\sum_{i=1}^{p} Q_i\big(t,x(\delta_-(s,t))\big)+\Phi_A(t,t_0)\left(x_0-\sum_{i=1}^{p} Q_i\big(t_0,x(\delta_-(s,t_0))\big)\right)$$
$$+\Phi_A(t,t_0)\int_{t_0}^{t}\Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u,x(\delta_-(s,u))\big)+\int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\right]\Delta u. \quad (6)$$

Since $x\big(\delta_+^T(t_0)\big)=x(t_0)=x_0$, (6) implies

$$x(t_0)-\sum_{i=1}^{p} Q_i\big(t_0,x(\delta_-(s,t_0))\big)=\Phi_A\big(\delta_+^T(t_0),t_0\big)\Big(x_0-\sum_{i=1}^{p} Q_i\big(t_0,x(\delta_-(s,t_0))\big)\Big)$$
$$+\Phi_A\big(\delta_+^T(t_0),t_0\big)\int_{t_0}^{\delta_+^T(t_0)}\Phi_A^{-1}(\sigma(u),t_0)\Big[A(u)\sum_{i=1}^{p} Q_i\big(u,x(\delta_-(s,u))\big) \quad (7)$$
$$+\int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\Big]\ \Delta u$$

substituting (7) into (6) yields

$$\begin{aligned}x(t) &= \sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big) + \Phi_A(t,t_0)\big(I - \Phi_A(\delta_+^T(t_0),t_0)\big)^{-1}\Phi_A(\delta_+^T(t_0),t_0)\\ &\times \int_{t_0}^{\delta_+^T(t_0)} \Phi_A^{-1}(\sigma(u),t_0)\Big[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\,\Big]\Delta u \qquad (8)\\ &+\Phi_A(t,t_0)\int_{t_0}^{t} \Phi_A^{-1}(\sigma(u),t_0)\Big[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\,\Big]\Delta u.\end{aligned}$$

In order to show that (8) is equivalent to (4) we use:

$$\begin{aligned}\big(I - \Phi_A(\delta_+^T(t_0),t_0)\big)^{-1} &= \big(\Phi_A(\delta_+^T(t_0),t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)\big)^{-1}\\ &= (\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1}\Phi_A^{-1}(\delta_+^T(t_0),t_0)\end{aligned}$$

to get

$$\begin{aligned}x(t) &= \sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big) + \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1}\Phi_A^{-1}(\delta_+^T(t_0),t_0)\Phi_A(\delta_+^T(t_0),t_0)\\ &\times \int_{t_0}^{\delta_+^T(t_0)} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\,\right]\Delta u\\ &+\Phi_A(t,t_0)\int_{t_0}^{t} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\,\right]\Delta u,\end{aligned}$$

Furthermore, we have the equality that follows:

$$\begin{aligned}x(t) &= \sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big) + \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1}\\ &\times\Bigg[\int_{t_0}^{\delta_+^T(t_0)} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\,\right]\Delta u\\ &+\Phi_A^{-1}(\delta_+^T(t_0),t_0)\int_{t_0}^{t} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\,\right]\Delta u\\ &-\int_{t_0}^{t} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s \quad \Delta u\right]\end{aligned}$$

thus, $x(t)$ can be stated as follows

$$\begin{aligned}x(t) &= \sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big) + \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1}\\ &\times\Bigg[\int_{t}^{\delta_+^T(t_0)} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,t))\big) + \int_{-\infty}^{t}\Big(D(t,s)f\big(x(\delta_-(s,t))\big)\Big)\Delta s\,\right]\Delta u\\ &+\Phi_A^{-1}(\delta_+^T(t_0),t_0)\int_{t_0}^{t} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big) + \int_{-\infty}^{t}\Big(D(t,s)f\big(x(\delta_-(s,t))\big)\Big)\Delta s\,\right]\Delta u\Bigg].\end{aligned}$$

If we let $u = \delta_-^T(\hat{u})$, we get

$$\begin{aligned}x(t) &= \sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big) + \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1} \\ &\times \Bigg[\int_t^{\delta_+^T(t_0)} \Phi_A^{-1}(\sigma(u),t_0)\Bigg[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s \Bigg]\Delta u \\ &+\Phi_A^{-1}(\delta_+^T(t_0),t_0)\Bigg[\Bigg[\int_{\delta_+^T(t_0)}^{\delta_+^T(t)} \Phi_A^{-1}(\sigma(\delta_-^T(\hat{u})),t_0)\Bigg[A(\delta_-^T(\hat{u}))\sum_{i=1}^{p} Q_i\left(\delta_-^T(\hat{u}), x\big(\delta_-(s,\delta_-^T(\hat{u}))\big)\right) \\ &+\Bigg[\int_{-\infty}^{\hat{u}}\Big(D(\hat{u},s)f\big(x(\delta_-(s,\delta_-^T(\hat{u})))\big)\Big)\Delta s \Bigg]\delta_-^{\Delta T}(\hat{u})\Delta u\Bigg]\Bigg]\end{aligned}$$

And

$$\begin{aligned}x(t) &= \sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big) + \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1} \\ &\times \Bigg[\int_t^{\delta_+^T(t_0)} \Phi_A^{-1}(\sigma(u),t_0)\Bigg[A(u)\sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big) + \int_{-\infty}^{t}\Big(D(t,s)f\big(x(\delta_-(s,t))\big)\Big)\Delta s\,] \Bigg]\Delta u \\ &+\Phi_A^{-1}(\delta_+^T(t_0),t_0)\int_{\delta_+^T(t_0)}^{\delta_+^T(t)} \Phi_A^{-1}(\sigma(\delta_-^T(\hat{u})),t_0)\Bigg[A(\hat{u})\sum_{i=1}^{p} Q_i\big(\hat{u}, x(\delta_-(s,t))\big) + \int_{-\infty}^{\hat{u}}\Big(D(\hat{u},s)f\big(x(\delta_-(s,\hat{u}))\big)\Big)\Delta s \Bigg]\Delta u\Bigg].\end{aligned}$$

Since

$$\Phi_A^{-1}(\delta_+^T(t_0),t_0)\Phi_A^{-1}(\sigma(\delta_-^T(\hat{u})),t_0) = \Phi_A^{-1}(\delta_+^T(t_0),t_0)\Phi_A^{-1}(\delta_-^T(\sigma(\hat{u})),t_0)$$

we have

$$\begin{aligned}\Phi_A\big(t_0,\delta_+^T(t_0)\big)\Phi_A\big(t_0,\delta_-^T(\sigma(\hat{u}))\big) &= \Phi_A\big(t_0,\delta_+^T(t_0)\big)\Phi_A\big(\delta_+^T(t_0),\sigma(\hat{u})\big) \\ &= \Phi_A\big(t_0,\sigma(\hat{u})\big).\end{aligned}$$

Substituting the final equality into the previous expression yields the following result:

$$\begin{aligned}x(t) = \sum_{i=1}^{p} &Q_i\big(t, x(\delta_-(s,t))\big) + \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - 1)^{-1} \\ &\times \Bigg[\int_t^{\delta_+^T(t)} \Phi_A^{-1}(\sigma(u),t_0)\Bigg[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s\Bigg]\Delta u\Bigg],\end{aligned}$$

as required. We now present Krasnoselskii's fixed point theorem, which will serve as a key tool in establishing the existence of a periodic solution

Theorem 6 (Krasnoselskii)

Let M be a closed convex nonempty subset of a Banach space $(B, \|\cdot\|)$. Suppose that B and C maps M into B such that

1. $x, y \in M$ implies $Bx + Cy \in M$.
2. C is compact and continuous.
3. B is a contraction mapping.

Then there exists $z \in M$ with $z = Bz + Cz$.

In preparation for the next result define the mapping H by

$$(H\varphi)(t) = (B\varphi)(t) + (C\varphi)(t)$$

where

$$(B\varphi)(t) := \sum_{i=1}^{p} Q_i\big(t, x(\delta_-(s,t))\big)$$

and

$$(C\varphi)(t) \quad := \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1} \times$$

$$\left[\int_t^{\delta_+^T(t)} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s \quad\right]\Delta u\right] \tag{11}$$

Lemma 4

Assume that conditions (a)–(b) and (2) are satisfied. Define the operator C as in Equation (11). Suppose further that there exist positive constants E_1, E_2, E_3, and N such that the following inequalities or properties hold:

$$\sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,y)| \le E_1\|x-y\|, |f(x) - f(y)| \le E_2\|x-y\|, \int_{-\infty}^{t} |D(t,u)|\Delta u \le E_3, \tag{12}$$

and

$$r(\delta_+^T(t_0) - t_0)(\|A\|E_1 + E_2E_3) \le N \tag{13}$$

hold, then

1. $\|(C\Phi)()\cdot\| \le r(\delta_+^T(t_0) - t_0)\left\|A(.)\sum_{i=1}^{p} Q_i(.,x(\delta_-(s,).)) + \int_{-\infty}\Big(D(.,s)f\big(x(\delta_-(s,).)\big)\Big)\Delta s\right\|$

   where

$$r = \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\left(\max_{u\in[t,\delta_+^T(t)]_{\mathbb{T}}} |\Phi_A(\sigma(u),t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)\Phi_A^{-1}(t,t_0)|^{-1}\right). \quad (14)$$

2. C is continuous and compact.

Proof

Define the operator C as given in Equation (11). It can then be expressed in the following equivalent form

$$(C\varphi)(t) \quad := \Phi_A(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1}$$

$$\times\left[\int_t^{\delta_+^T(t)} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i\big(u, x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s \quad\right]\Delta u\right]$$

Since $(C\varphi)(t) \in P_T$, we have

$$\|(C\varphi)(.)\| = \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\int_t^{\delta_+^T(t)} [\Phi_A(\sigma(u),t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)\Phi_A^{-1}(t,t_0)]^{-1}\left[\sum_{i=1}^{p} Q_i\big(u,x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s \quad\right]\Delta u$$

$$\le \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}\left(\max_{u\in[t,\delta_+^T(t)]_{\mathbb{T}}} |[\Phi_A(\sigma(u),t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)\Phi_A^{-1}(t,t_0)]^{-1}|\right)$$

$$\times \max_{t\in[t_0,\delta_+^T(t_0))}\int_{\mathbb{T}}\int_{t_0}^{\delta_+^T(t_0)}\left|A(u)\sum_{i=1}^{p} Q_i\big(u,x(\delta_-(s,u))\big) + \int_{-\infty}^{u}\Big(D(u,s)f\big(x(\delta_-(s,u))\big)\Big)\Delta s \quad\right|\Delta u$$

$$\le r(\delta_+^T(t_0) - t_0)\left\|A(.)\sum_{i=1}^{p} Q_i\big(.,x(\delta_-(s,.))\big) + \int_{-\infty}^{.}\Big(D(.,s)f\big(x(\delta_-(s,.))\big)\Big)\Delta s \quad\right\|.$$

This concludes the proof of part (i). To establish the continuity of the operator C, suppose that φ and ψ are elements of P_TGiven any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} \| (C\varphi)(.) - (C\psi)(.) \| \quad &\le r \int_{t_0}^{\delta_+^T(t_0)} [\| A \| E_1 \| \varphi - \psi \| + (E_2E_3) \| \varphi - \psi \|]\Delta u \\ &\le r(\delta_+^T(t_0) - t_0)(\| A \| E_1 + E_2E_3) \| \varphi - \psi \| < \varepsilon \end{aligned}$$

By selecting $\delta = \varepsilon/N$, we establish the continuity of the operator C

To demonstrate that C is compact, we define the set $D := \{\phi \in P_T : \|\phi\| \le R\}$,where R >0 is a fixed constant. Consider a sequence $\{\phi_n\}$, of T-periodic functions in shifts such that $\{\phi_n\} \in D$. Furthermore, using estimates (12) and (13), we obtain the following bounds

$$\begin{aligned} \sum_{i=1}^{p} |Q_i(t,x)| &= \sum_{i=1}^{p} |(Q_i(t,x) - Q_i(t,0) + Q_i(t,0))| \\ &\le \sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,0)| + |Q_i(t,0)| \\ &\le E_1\|x\| + \alpha \end{aligned}$$

and

$$\begin{aligned} |f(x)| &= |(f(x) - f(0) + f(0))| \\ &\le |f(x) - f(0)| + |f(0)| \\ &\le E_3\|x\| + \beta \end{aligned}$$

where $\alpha = |Q_i(t,0)|$ and $\beta = |f(0)|$. If we consider $\|(C\phi_n)().\|$, we have

$$\|(C\phi_n)(.)\| \le r(\delta_+^T(t_0) - t_0)[\|A\|(E_1\|\phi_n\| + \alpha) + E_2E_3\|\phi_n\| + \beta]$$

where r is as in (14). Since $\{\phi_n\} \in D$, we obtain $\|(C\phi_n)().\| \le L$, where

$$L = r(\delta_+^T(t_0) - t_0)[\|A\|(E_1\|\phi_n\| + \alpha) + E_2E_3\|\phi_n\| + \beta].$$

We now compute the delta derivative$(C\phi_n)^{\Delta}(t)$ and demonstrate that the sequence $\{C\phi_n\}$ is uniformly bounded.

$$\begin{aligned} (C\phi_n)^{\Delta}(t) &= \Phi_A^{\Delta}(t,t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1} \times \int_t^{\delta_+^T(t_0)} \Phi_A^{-1}(\sigma(u),t_0)\left[A(u)\sum_{i=1}^{p} Q_i(u,\phi_n(\delta_-(s,u))) + \int_{-\infty}^{u} (D(u,s)\right. \\ &+\Phi_A(\sigma(t),t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1} \times \left[\Phi_A^{-1}(\sigma(\delta_+^T(t)),t_0)\left[A(\delta_+^T(t))\sum_{i=1}^{p} Q_i\left(\delta_+^T(t),\phi_n(\delta_-(s,\delta_+^T(t)))\right)\right] + \int_{-\infty}^{t} (D(u,s)\right. \\ &-\Phi_A^{-1}(\sigma(t),t_0)\left[A(t)\sum_{i=1}^{p} Q_i(t,\phi_n(\delta_-(s,t))) + \int_{-\infty}^{t} \left(D(t,s)f(\phi_n(\delta_-(s,t)))\right)\Delta s\right]. \end{aligned}$$

This along with (a-c) and

$$\Phi_A^{\Delta}(t,t_0) = A(t)\Phi_A(t,t_0),$$

implies

$$\begin{aligned} (C\varphi_n)^{\Delta}(t) \quad &= A(t)(C\varphi_n)(t) + \Phi_A(\sigma(t),t_0)(\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)^{-1} \\ &\times\left[\left(\Phi_A^{-1}(\sigma(\delta_+^T(t)),t_0) - \Phi_A^{-1}(\sigma(t),t_0)\right)\left[A(t)\sum_{i=1}^{p} Q_i(t,\varphi_n(\delta_-(s,t))) + \int_{-\infty}^{t} \left(D(t,s)f(\varphi_n(\delta_-(s,t)))\right)\Delta s\right]\right]. \end{aligned}$$

Substituting

$$\Phi_A^{-1}(\sigma(\delta_+^T(t)),t_0) - \Phi_A^{-1}(\sigma(t),t_0) = (\Phi_A^{-1}(\delta_+^T(t_0),t_0) - I)\Phi_A^{-1}(\sigma(t),t_0)$$

in (15), we obtain

$$(C\varphi_n)^{\Delta}(t) = A(t)(C\varphi_n)(t) + A(t)\sum_{i=1}^{p} Q_i(t,\varphi_n(\delta_-(s,t))) + \int_{-\infty}^{t} \left(D(t,s)f(\varphi_n(\delta_-(s,t)))\right)\Delta s \quad .$$

Therefore, the sequence $(C\phi_n)^{\Delta}(t)$is bounded. This implies that $\{C\phi_n\}$ is not only uniformly bounded but also equicontinuous. Consequently, by the Arzelà–Ascoli theorem, the image $C(D)$ is relatively compact, and hence compact in the space P_T

Lemma 5

Let the operator B be defined as in Equation (10). If condition (12) holds with E_1<ζ<1, then B is a contraction mapping.

Proof

Assume that B is given by Equation (10). Then, for any $\phi, \psi \in P_T$, we have the following estimate:

$$\begin{aligned}\|(B\varphi)(.) - (B\psi)(.)\| &= \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} |(B\varphi)(t) - (B\psi)(t)| \\ &= \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \sum_{i=1}^{p} \left|Q_i\big(t,\varphi(\delta_-(s,t))\big) - Q_i\big(t,\psi(\delta_-(s,t))\big)\right| \\ &\le E_1\|\varphi-\psi\| \\ &< \zeta\|\varphi-\psi\|\end{aligned}$$

This shows B is a contraction mapping with contraction constant ζ.

Theorem 7

Suppose that all the hypotheses of Lemma 2.1 are satisfied. Let r be defined as in Equation (14), and set $\alpha := \|Q_i(t,0)\|$ and $\beta := \|f(0)\|$. Let $J > 0$ be a constant satisfying

$$E_1 J + \alpha + r(\delta_T^+(t_0) - t_0)[\|\mathbf{A}\|(\alpha + E_1 J) + (E_2E_3)J + \beta] \le J.$$

Then, Equation (1) admits at least one solution in the set

$$M := \{\phi \in P_T : \|\phi\| \le J\}.$$

Proof. From Lemma 4, we establish that:

- The operator C is both continuous and compact on P_T.
- The operator B acts as a contraction mapping on P_r

To invoke Krasnoselskii's fixed point theorem, we must verify the bound $\|B\psi + C\phi\| \le J$ for all $\phi, \psi \in M$. For arbitrary $\phi, \psi \in M$, we derive:

$$\begin{aligned}\|B\psi(0) + C\phi(0)\| \le\; & E_1\|\psi\| + \alpha \\ & + r\int_{i_0}^{\delta T}(t_0) \\ & + (E_2E_3)\|\phi\| + \beta\|(\alpha + E_1\|\phi\|) \\ \le\; & E_1 J + \alpha \\ & + r(\delta_+^T(t_0) - t_0)[\|A\|(\alpha + E_1 J) \\ & + (E_2E_3)J + \beta] \\ \le\; & J\end{aligned}$$

This demonstrates $\|B\psi + C\phi\| \le J$ uniformly for all $\phi, \psi \in M$. Krasnoselskii's theorem guarantees existence of $z \in M$ satisfying:

$$z = Bz + Cz$$

This fixed point z represents a completing the proof.

Theorem 8

In addition to the assumptions of Lemma 2.1, suppose further that

$$E_1 + r(\delta_T^+(t_0) - t_0)(\|\mathbf{A}\|E_1 + E_2E_3) < 1.$$

Then, Equation (1) possesses a unique solution that is T-periodic in shifts δ_T^+.

Proof. Define the operator $H = B + C$. Then,

$$\|H\phi - H\psi\| \leq [E_1 + r(\delta_T^+(t_0) - t_0)(\|\mathbf{A}\|E_1 + E_2E_3)]\|\phi - \psi\|,$$

which is a strict contraction. Hence, by the Banach fixed point theorem, H has a unique fixed point in P_T.

. The proof is complete. The following result is a generalization of (Raffoul, 2005).

Corollary 3

Assume that conditions (a)-(c) and (2) hold. Let $\alpha := \|Q_i(t,0)\|$ and $\beta := \|f(0)\|$. Suppose that there exist positive constants E_1^*, E_2^*, E_3^* such that certain bounds or Lipschitz-type conditions are satisfied. Then, under these assumptions, the existence of a periodic solution can be established.

$$\sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,y)| \leq E_1^*\|x-y\|, |f(x) - f(y)| \leq E_2^*\|x-y\|, \int_{-\infty}^{t} |D(t,u)|\Delta u \leq E_3^*$$

and

$$E_1^*J + \alpha + r(\delta_+^T(t_0) - t_0)[\|A\|(\alpha + E_1^*J) + (E_2^*E_3^*)J + \beta] \leq J$$

holds for all $x, y, z, w \in M$. Then (1) has a solution in M. Moreover, if

$$E_1^* + r(\delta_+^T(t_0) - t_0)(\|A\|E_1^* + E_2^*E_3^*) < 1$$

then the solution in M is unique.

$$\sum_{i=1}^{p} |Q_i(t,x) - Q_i(t,y)| \leq E_1^*\|x-y\|, |f(x) - f(y)| \leq E_2^*\|x-y\|, \int_{-\infty}^{t} |D(t,u)|\Delta u \leq E_3^*$$

and

$$E_1^*J + \alpha + r(\delta_+^T(t_0) - t_0)[\|A\|(\alpha + E_1^*J) + (E_2^*E_3^*)J + \beta] \leq J$$

holds for all $x, y, z, w \in M$. Then (1) has a solution in M. Moreover, if

$$E_1^* + r(\delta_+^T(t_0) - t_0)(\|A\|E_1^* + E_2^*E_3^*) < 1$$

then the solution in M is unique.

CONCLUSION

This study investigated the existence of periodic solutions for neutral nonlinear dynamic systems with delay on time scales by introducing a novel periodicity framework based on shift operators. By generalizing the concept of periodicity beyond the traditional additive setting, this work provides a more flexible foundation for analyzing a broader class of dynamic models, especially those arising in biological and applied sciences. We redefined periodicity using shift operators, allowing the treatment of dynamic equations on time scales that do not conform to classical additive structures. This approach overcomes the limitations of earlier methodologies and expands the analytical reach to more general time domains. Utilizing Krasnoselskii's fixed point theorem in conjunction with Floquet theory, we established sufficient conditions for the existence of periodic solutions. The formulation relies on integral representations and operator-theoretic techniques that enable rigorous handling of systems with delay. The results have direct relevance to delayed population dynamics and hematopoiesis models, providing effective tools to characterize and predict periodic behaviors in systems with memory and feedback mechanisms. In summary, this study contributes to the deeper understanding of periodicity in dynamic systems on general time scales and sets the stage for future work on hybrid, discrete-continuous, and other non-standard systems. The framework developed has the potential to enhance modeling accuracy in disciplines such as biology, engineering, and economics, where periodic phenomena play a critical role.

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