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Some Types of near Normality Based on Soft Simply Open Sets

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Abstract

This study investigates the fundamental properties and applications of soft simply open sets within soft topological spaces. We establish novel theoretical frameworks by:

1- Characterizing their relations to soft separation axioms (particularly near normality).

2- Demonstrating their duality through soft, simply interior (closure) operations, and proving their efficacy in handling uncertain data structures.

The results highlight how soft simply open sets bridge theoretical topology with computational applications, offering new tools for complex data analysis. Future directions include extensions to soft compactness and connection with fuzzy neural networks.

Keywords: Soft Sets, Soft Topology, Soft Simply Open Set, Soft Simply Normal, Soft Simply-Exterior, Soft Simply-Boundary.

INTRODUCTION

Soft set notion was introduced, for the first time, by (Molodtsov 2017) as mathematical models in order to deal with vagueness and sets to solve complicated problems in fields such as computer sciences, medical sciences, economics, and engineering, among others. Moreover, he studied the main points of this theory and its properties.

Shabir et al. (Shabir 2011) introduced and studied the concept of soft topological spaces, which are defined over an initial universe with a fixed set of parameters.

After that, many researchers (Arockiaran, Chen 2013; Hussain, Min, Nazmul 2013; Zorlutuna 2013) added several notions and concepts towards (Maji 2003, Min 2011; Molodtsov 1999; Zorlutuna 2012) the properties of soft topological spaces. Recently, quite a number of generalizations of the class of soft open sets in a soft topological space have been considered.

Shabir and Naz introduced the new notion of soft topological space. (Shabir 2011). Then, many authors (Zorlutuna 2012) studied some of main concepts and properties of soft topology. Chen introduced the new soft near open sets, so called soft semi-open sets. (Chen 2013).

Many scientists, including (Al Ghour, Hamed 2020; Al-Shami 2020; Cetkin, Aygun 2020; Hussain 2019) have conducted various and very important studies to study simple open smooth groups in topological spaces, where they studied their properties and their relationship to various topics in mathematics and other fields. They presented multiple theories and proofs that we did not address



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in this paper because these topics are new and branched, but we can connect them to what we have studied, and this will be done in new papers.

In the present paper, we shall define new concepts in soft topological space, such as “soft simply open sets, soft simply derived, soft simply dense and soft simply residual.

1- Preliminaries:

Definition 1.1 (Molodtsov 1999)

A pair (F, E) is called a soft set over X , where F is a mapping given by $F: E \rightarrow P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe. For a particular $e \in E$, $F(e)$ may be considered the set of e -approximate elements of the soft set (F, E) .

Definition 1.2 (Maji 2003)

For two soft sets (F, A) and (G, B) over a common universe X , we say that (F, A) is a soft subset of (G, B) if:

i- $A \subseteq B$.

ii- $\forall e \in A, F(e) \subseteq G(e)$ are identical approximations. We write $(F, A) \subseteq (G, B)$. (F, A) is said to be a soft super set of (G, B) , if (G, B) is a soft subset of. We denote it by $(F, A) \supseteq (G, B)$.

Definition 1.3 (Ali 2009)

The relative complement of a soft set (F, E) is denoted by $(F, E)^c$ and is

defined by $(F, E)^c = (F^c, E)$ where $F^c: E \rightarrow P(X)$ is a mapping given by $F^c(e) = X - F(e)$ for all $e \in E$.

Definition 1.4 (Shabir 2011)

Let τ be the collection of soft sets over X , then τ is said to be a soft topology on X if:

1- $\emptyset, X$ belong to τ .

2- The union of any member of soft sets belongs to τ .

3- The intersection of any two soft sets belongs to τ .

The triplet (X, τ, E) is called a soft topological space over X .

The members of τ are said to be soft open sets in X , we will denote all soft open sets (resp. soft closed sets) in X as $SO(X)$ (resp. $SC(X)$).

Definition 1.5 (Maji 2003)

A soft set (F, A) over X is said to be:

i- Null soft set denoted by $\emptyset$, if $\forall e \in A, F(e) = \emptyset$.

ii- Absolute soft set denoted by $\tilde{A}$, if $\forall e \in A, F(e) = X$.

Definition 1.6 (Maji 2003)

For two soft sets (F, A) and (G, B) over a common universe set X , we define:

1- Union of two soft sets of (F, A) and (G, B) is the soft set (H, C) , where $C = A \cup B$, and $\forall e \in C$,

then $H(e) = \{F(e), \text{if } e \in A - B \text{ or } G(e) \text{ if } e \in B - A \text{ or } F(e) \cup G(e), \text{if } e \in A \cap B\}$, we write $(F, A) \cup (G, B) = (H, C)$.

2- Intersection of (F, A) and (G, B) is the soft set (H, C) , where $C = A \cap B$, and $\forall e \in C, H(e) = F(e) \cap G(e)$, we write $(F, A) \cap (G, B) = (H, C)$.

Definition 1.7 (Bayramov 2013)

Let (F, E) be a soft set in X . The soft set (F, E) is called a soft point, denoted by (x_e, E) or x_e , if for the element $e \in E, F(e) = \{x\}$ and $F(e^c) = \emptyset$, for all $e^c \in E - \{e\}$.

Definition 1.8 (Zorlutuna 2012)

The soft point x_e is said to belong to the soft set (G, E) , denoted by $x_e \in (G, E)$, if for the element $e \in E, F(e) \subseteq G(e)$.

Definition 1.9 (El-sayed 2017)

A soft subset (F, E) of a soft topological space (X, τ, E) is called

1- Soft nowhere dense if $\text{Sint}(Scl(A, E)) = \emptyset$.

2- Soft simply open set if $(F, A) = (G, E) \cup (V, E)$ where (G, E) is soft open set and (V, E) is soft nowhere dense set.

We shall denote the class of soft simply open, (resp. soft simply closed, soft nowhere dense,) sets of a universe set X by $SS^MO(X, E)$ (resp. $SS^Mc(X, E), SN(X, E)$).

Definition 1.10 (Hussain 2014)

Let (X, τ, E) be a soft topological space over X , (G, E) be a soft set over X and $x \in X$, then x is said to be a soft simply-interior point of (G, E) , if there exists a soft simply open set (F, E) such that $x \in (F, E) \subseteq (G, E)$.

Definition 1.11 (Hussain 2014)

Let (X, τ, E) be a soft topological space over X , then the soft simply exterior of the soft set (F, E) over X is denoted by $SSext(F, E)$ and defined as

$SSext(F, E) = SSint(X - (F, E))$. Thus x is called a soft simply exterior point of (F, E) if there exists a soft simply open set (G, E) such that $x \in (G, E) \subseteq (F, E)$, we observe that $SSext(F, E)$ is the largest soft simply open set contained in (F, E) .

2- Some Types of Near Normality Based on Soft Simply Open Sets

In this section, we introduce some types of normality in soft topological space (X, τ, E) based on soft simply open, also we study their properties.

Definition 2.1

A soft topological space (X, τ, E) is called soft simply normal (S^M -normal) if

$\forall (G_1, E), (G_2, E) \in \mathcal{C}(X), (G_1, E) \cap (G_2, E) = \emptyset, \exists (F_1, E), (F_2, E) \in SS^MO(X),$

$(F_1, E) \cap (F_2, E) = \emptyset, \exists (G_1, E) \subseteq (F_1, E) \text{ \& } (G_2, E) \subseteq (F_2, E).$

Definition 2.2

A soft topological space $(X, \tilde{\tau}, E)$ is said to be soft simply regular ($SS^M R$) if
 $\forall (A, E) \in SC(X), x \notin (A, E), \exists (U, E), (V, E) \in SS^M O(X), (U, E) \tilde{\cap} (V, E) = \tilde{\emptyset}$
 $\exists x \in (U, E) \& (A, E) \subseteq (V, E).$

Definition 2.3

A soft subset (A, E) of a soft topological space $(X, \tilde{\tau}, E)$ is called soft simply open (for short, SS^M -open) set if
 $(A, E) \in \{X, \emptyset, (G, E) \tilde{\cup} (N, E) : (G, E) \text{ is a proper open set and } (N, E) \text{ is a nowhere dense set}\}.$

The family of all soft simply open sets is denoted by $SS^M O(X)$, the complement of a soft simply open set is said to be soft simply closed (for short, SS^M -closed) set and denoted by $SS^M C(X)$.

Remark 2.1

In general, the intersection of finite number of SS^M -open sets in a soft topological space $(X, \tilde{\tau}, E)$ is not SS^M -open set.

Example 2.1

Consider a soft topological space $(X, \tilde{\tau}, E)$, where $X = \{x_1, x_2\}$ is the universal set, $E = \{e_1, e_2\}$ is the set of parameters, $\tilde{\tau}$ is the soft topological on X . Define two soft simply open sets A & B as follows:

$A = \{(e_1, \{x_1\}), (e_2, \{x_2\})\}$, $B = \{(e_1, \{x_2\}), (e_2, \{x_1\})\}$. Here, both A & B are soft simply open sets in the soft topology $\tilde{\tau}$. So $A \tilde{\cap} B = \{(e_1, \{x_1\}) \tilde{\cap} \{x_2\}, (e_2, \{x_2\}) \tilde{\cap} \{x_1\}\} = \{(e_1, \tilde{\emptyset}), (e_2, \tilde{\emptyset})\}$.

This set is not a soft simply open set because it contains empty sets for all parameters, which contradicts the definition of soft simply open set in most soft topological spaces.

Definition 2.4

Let $(X, \tilde{\tau}, E)$ be any soft topological space, and $(A, E) \subseteq X$, we define the SS^M -boundary of (A, E) as follows $SS^M b(A, E) = SS^M cl(A, E) \tilde{\cap} SS^M cl(X - (A, E))$.

Also, soft simply boundary of $(A, E) = \overline{(A, E)} - \text{int}(A, E)$ where $\text{int}(A, E)$ is the soft interior of (A, E) & (A, E) is soft simply set.

Remark 2.2

For any subset (A, E) of a soft topological $(X, \tilde{\tau}, E)$, we have $SS^M b(A, E) \subseteq b(A, E)$ and $SS^M b(A, E) \subseteq Sab(A, E)$.

Example 2.2

Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ be parameter, and $\tau = \{X, \tilde{\emptyset}, (F_1, E), (F_2, E)\}$, where
 $(F_1, E) = \{(e_1, \{x_1\}), (e_2, \tilde{\emptyset})\}$, $(F_2, E) = \{(e_1, X), (e_2, \{x_2\})\}$. This is topology over X ,
let $(A, E) = \{(e_1, \{x_1\}), (e_2, \{x_2\})\} \Rightarrow$
 $bd(A, E) = cl(A, E) \tilde{\cap} cl(X - (A, E)) = (A, E) \tilde{\cap} \{(e_1, \{x_1\}), (e_2, \{x_2\})\} = (A, E) \tilde{\cap} X$
 $= (A, E)$, $\alpha b(A, E) = cl(\alpha \text{int}(A, E)) \tilde{\cap} cl(X - \alpha \text{int}(A, E))$, but $\alpha \text{int}(A, E) = (F_1, E)$
 $= \{(e_1, \{x_1\}), (e_2, \tilde{\emptyset})\}$, $cl(\alpha \text{int}(A, E)) = (F_1, E)$, $cl(X - \alpha \text{int}(A, E)) = X$, thus
 $bd(A, E) = (F_1, E) \tilde{\cap} X = (F_1, E)$. Also, $SSb(A, E) = SScl(A, E) \tilde{\cap} SScl(X - (A, E))$ but
 $SScl(A, E) = (A, E)$ and $SScl(X - (A, E)) = X$, so $SSb(A, E) \subseteq (A, E) = b(A, E)$ and since

$SScl(A, E) \cong \alpha cl(A, E)$, we get $SSb(A, E) \cong \alpha b(A, E) = (F_1, E)$, thus $SSb(A, E) = \tilde{\emptyset}$ and $SSb(A, E) \cong \alpha b(A, E) \cong b(A, E)$.

Theorem 2.1

For any soft topological space $(X, \tilde{\tau}, E)$ and $(A, E), (B, E) \subseteq X$. Then the following statement hold:

- 1- $SS^M b(A, E) = SS^M b(X - (A, E))$.
- 2- $SS^M b(A, E) = SS^M cl(A, E) - SS^M int(A, E)$.
- 3- $SS^M b(A, E) \tilde{\cap} SS^M int(A, E) = \tilde{\emptyset}$.
- 4- $SS^M b(A, E) \tilde{\cup} SS^M int(A, E) = SS^M cl(A, E)$.

Proof

1- By definition, the soft simply boundary of (A, E) is

$SS^M b((A, E)) = SS^M cl((A, E)) - SS^M int((A, E))$. Similarly, the soft simply boundary of $X - ((A, E))$ is $SS^M b(X - (A, E)) = SS^M cl(X - (A, E))$, in soft topology the soft simply closure of (A, E) is related to the soft simply interior of its complement:
 $SS^M cl((A, E)) = X - SS^M int(X - (A, E))$, Thus $SS^M b((A, E)) = SS^M b(X - (A, E))$.

2- Since $SS^M cl(X - (A, E)) = X - SS^M int(A, E)$, then
 $SS^M b(A, E) = SS^M cl(A, E) \tilde{\cap} SS^M cl(X - (A, E)) = SS^M cl(A, E) \tilde{\cap} [X - SS^M int(A, E)]$
 $= SS^M cl(A, E) - [SS^M cl(A, E) \tilde{\cap} SS^M int(A, E)] = SS^M cl(A, E) - SS^M int(A, E)$.

3- By definition, $SS^M b((A, E)) = SS^M cl((A, E)) - SS^M int((A, E))$.
 This means $SS^M b((A, E))$ consists of point in $SS^M cl((A, E))$ that are not in $SS^M int((A, E))$.
 Therefore, $SS^M b((A, E))$ and $SS^M int((A, E))$ are disjoint by construction:
 $SS^M b((A, E)) \tilde{\cap} SS^M int((A, E)) = \tilde{\emptyset}$.

4- By definition, $SS^M b((A, E)) = SS^M cl((A, E)) - SS^M int((A, E))$, i.e.
 $SS^M b((A, E)) \tilde{\cup} SS^M int((A, E)) = SS^M cl((A, E))$. The union of $SS^M b((A, E))$ and $SS^M int((A, E))$ covers all points in $SS^M cl((A, E))$ and they are disjoint (from 3), thus
 $SS^M b((A, E)) \tilde{\cup} SS^M int((A, E)) = SS^M cl((A, E))$, then
 $SS^M b(A, E) \tilde{\cup} SS^M int(A, E) = SS^M cl(A, E)$.

Theorem 2.2

For any soft topological space (X, τ, E) and $(A, E) \subseteq X$, the following statement holds:

- 1- $(A, E) \cong SS^M O(X)$ iff $(A, E) \tilde{\cap} SS^M b((A, E)) = \tilde{\emptyset}$.
- 2- $(A, E) \cong SS^M C(X)$ iff $SS^M b((A, E)) \subseteq (A, E)$.
- 3- $(A, E) \cong SS^M O(X) \tilde{\cap} SS^M cl(X)$ iff $SS^M b((A, E)) = \tilde{\emptyset}$.

Proof

1- $\Rightarrow$ Assume that (A, E) is soft simply open, by definition, the soft simply boundary of (A, E) is. Since (A, E) is soft simply open, $SS^M int((A, E)) = (A, E)$, thus,

$SS^M b((A, E)) = SS^M cl((A, E)) - (A, E)$, i.e. $SS^M b((A, E))$ contains no points from (A, E) , so $(A, E) \tilde{\cap} SS^M b(A, E) = \tilde{\emptyset}$.

$\Leftarrow$ Assume that $(A, E) \tilde{\cap} SS^M b(A, E) = \tilde{\emptyset}$, by definition

$SS^M b(A, E) = SS^M cl(A, E) - SS^M int(A, E)$. If $(A, E) \tilde{\cap} SS^M b(A, E) = \tilde{\emptyset}$, then (A, E) contains no points from $SS^M cl(A, E) - SS^M int(A, E)$, this implies $(A, E) \tilde{\subseteq} SS^M int(A, E)$. Since $SS^M int(A, E)$ is the largest soft simply open set contained $int(A, E)$, we have $(A, E) = SS^M int(A, E)$, thus (A, E) is soft simply open.

2- $\Rightarrow$ Assume (A, E) is soft simply closed, by definition the soft simply boundary of (A, E) is. Since (A, E) is soft simply closed, $SS^M cl(A, E) = (A, E)$, thus

$$SS^M b(A, E) = SS^M cl(A, E) - SS^M int(A, E), \text{ i.e. } SS^M b(A, E) \tilde{\subseteq} (A, E).$$

By definition $SS^M b(A, E) = SS^M cl(A, E) - SS^M int(A, E)$, if $SS^M b(A, E) \tilde{\subseteq} (A, E)$, then $SS^M cl(A, E) - SS^M int(A, E) \tilde{\subseteq} (A, E)$, this implies $SS^M cl(A, E) \tilde{\subseteq} (A, E)$. So since $(A, E) \tilde{\subseteq} SS^M cl(A, E)$ by definition, we have $SS^M cl(A, E) = (A, E)$, thus (A, E) is soft simply closed.

3- $\Rightarrow$ Assume that (A, E) is both soft simply open and soft simply closed. Since (A, E) is soft simply open, $SS^M int(A, E) = (A, E)$, and since (A, E) is soft simply closed, $SS^M cl(A, E) = (A, E)$. By definition, the soft simply boundary is we have $SS^M b(A, E) = SS^M cl(A, E) - SS^M int(A, E)$.

$\Leftarrow$ Assume that $SS^M b(A, E) = \tilde{\emptyset}$, so by definition

$$SS^M b(A, E) = SS^M cl(A, E) - SS^M int(A, E). \text{ If } SS^M b(A, E) = \tilde{\emptyset}, \text{ then}$$

$SS^M cl(A, E) \tilde{\subseteq} SS^M int(A, E)$, and since $SS^M int(A, E) \tilde{\subseteq} (A, E) = SS^M cl(A, E)$ thus (A, E) is both soft simply open and soft simply closed.

Remark 2.3

For any soft topological space (X, τ, E) and $(A, E) \tilde{\subseteq} X$, we have

$$Sext(A, E) \tilde{\subseteq} Saext(A, E) \tilde{\subseteq} SS^M ext(A, E).$$

proof

We need to show that $SS^M ext(A, E) \tilde{\subseteq} Ext(A, E)$, by definition, the soft simply interior of a set is the largest soft simply open set contained in it. Since every soft simply open set is also a soft open set, we have $SS^M int(X - (A, E)) \tilde{\subseteq} int(X - (A, E))$, therefore,

$$SS^M ext(A, E) = SS^M int(X - (A, E)).$$

Also, we show that $Ext(A, E) \tilde{\subseteq} SS^M ext(A, E)$, by definition, the soft interior of a set is the largest soft open set contained in it. Since every soft open set is also a soft simply open set, we have,

$$int(X - (A, E)) \tilde{\subseteq} SS^M int(X - (A, E)), \text{ therefore } Ext(A, E) = int(X - (A, E)),$$

then $SS^M ext(A, E) \tilde{\subseteq} Ext(A, E) \tilde{\subseteq} SS^M ext(A, E)$.

Example 2.3

Consider a soft topological space $(X, \tilde{\tau}, E)$, where: $X = \{x_1, x_2\}$, is the universal set, $E = \{e_1, e_2\}$ is the set of parameters, $\tilde{\tau} = \{\tilde{\emptyset}, X, (F_1, E), (F_2, E)\}$ is the soft topology, with:

$(F_1, E) = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$, $(F_2, E) = \{(e_1, \{x_2\}), (e_2, \{x_2\})\}$. Let (A, E) be a soft subset of (X, τ, E) defined as: $(A, E) = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$. The complement of (A, E) is

$(A^c, E) = \{(e_1, \{x_2\}), (e_2, \{x_2\})\}$, the soft simply interior of (A^c, E) is $SS^M int(A^c, E) = (A^c, E)$,

because (A^c, E) is soft simply open in this topology. Thus, $SS^M ext(A, E) = SS^M int(A^c, E) = (A, E)$, $Saint(A^c, E) = \tilde{\emptyset}$ because there is no soft alpha open set contained in (A^c, E) in this topology, Thus, $Sexta(A, E) = Saint(A^c, E) = \tilde{\emptyset}$. So that we have: $SS^M ext(A, E) = (A^c, E) = \{(e_1, \{x_2\}), (e_2, \{x_2\})\}$, and we have: $SS^M ext(A, E) \not\subseteq Sexta(A, E)$, because $SS^M ext(A, E)$ is nonempty while $Sexta(A, E)$ is empty then, $SS^M ext(A, E) \not\subseteq Sa(A, E)$.

Example 2.4

Consider a soft topological space $(X, \tilde{\tau}, E)$, where: $X = \{x_1, x_2\}$, is the universal set, $E = \{e_1, e_2\}$ is the set of parameters, $\tilde{\tau} = \{\tilde{\emptyset}, X, (F_1, E), (F_2, E)\}$ is the soft topology, with $(F_1, E) = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$, $(F_2, E) = \{(e_1, \{x_2\}), (e_2, \{x_2\})\}$. Let (A, E) be a soft subset of (X, τ, E) defined as $(A, E) = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$. The complement of (A, E) is $(A^c, E) = \{(e_1, \{x_2\}), (e_2, \{x_2\})\}$, the soft simply interior of (A^c, E) is $SS^M int(A^c, E) = (A^c, E)$, because (A^c, E) is soft simply open in this topology so, $SS^M ext(A, E) = SS^M int(A^c, E) = (A, E)$. The soft simply closure of (A, E) is $SS^M cl(A, E) = (A, E)$, because (A, E) is soft simply closed in this topology, and the soft simply interior of (A, E) is $SS^M int(A, E) = (A, E)$, because (A, E) is soft simply open in this topology. Thus, $SS^M b(A, E) = SS^M cl(A, E) - SS^M int(A, E)$, so we have: $SS^M b(A, E) = \tilde{\emptyset}$, and we observe that $SS^M ext(A, E) \tilde{\cap} SS^M b(A, E) = \tilde{\emptyset}$.

Theorem 2.3

For any soft topological space $(X, \tilde{\tau}, E)$ and $(A, E), (B, E) \subseteq X$, the following statements hold:

- 1- $SS^M ext(A, E) = SS^M int(X - (A, E))$.
- 2- $SS^M ext(A, E)$ is SS^M -open.
- 3- $SS^M ext(A, E) \tilde{\cap} SS^M bd(X - (A, E)) = \tilde{\emptyset}$.
- 4- $SS^M ext(A, E) \tilde{\cup} SS^M bd(X - (A, E)) = SS^M cl(X - (A, E))$.
- 5- The set $\{SS^M int(A, E), SS^M bd(A, E), SS^M ext(A, E)\}$ form a partition of X .
- 6- If $(A, E) \subseteq (B, E)$, then $SS^M ext(B, E) \subseteq SS^M ext(A, E)$.
- 7- $SS^M ext(A, E) \tilde{\cup} (B, E) \subseteq SS^M ext(A, E) \tilde{\cup} SS^M ext(B, E)$.
- 8- $SS^M ext((A, E) \tilde{\cap} (B, E)) \subseteq SS^M ext(A, E) \tilde{\cap} SS^M ext(B, E)$.
- 9- $SS^M ext(\tilde{\emptyset}) = X$, and $SS^M ext(X) = \tilde{\emptyset}$.

Proof

Obvious from definitions

Example 2.5

Let $X = \{x_1, x_2, x_3\}$ with the soft topology τ defined as $\tilde{\tau} = \{\tilde{\emptyset}, X, (A, E), (B, E)\}$, where $(A, E) = \{(x_1, E)\}$, $(B, E) = \{(x_2, E)\}$. The $SS^M ex(A, E) = \{(x_2, E), (x_3, E)\}$, and $SS^M ex(B, E) = \{(x_1, E), (x_3, E)\}$,

$SS^M ex(A, E) \tilde{\cup} SS^M ex(B, E) = \{(x_1, E), (x_2, E), (x_3, E)\}$. Now, consider $(A, E) \tilde{\cup} (B, E)$ is $\{(x_3, E)\}$. Here, $\{(x_3, E)\} \subseteq \{(x_1, E), (x_2, E), (x_3, E)\}$ showing that equality dose not hold.

Example 2.6

Using the same X and $\tilde{\tau}$ as above: $SS^M ex(A, E) = \{(x_2, E), (x_3, E)\}$,

$$SS^M ex(B, E) = \{(x_1, E), (x_3, E)\}, SS^M ex(A, E) \tilde{\cap} SS^M ex(B, E) = \{(x_3, E)\}.$$

Now $(A, E) \tilde{\cap} (B, E) = \emptyset$, $SS^M ex(\emptyset) = X = \{(x_1, E), (x_2, E), (x_3, E)\}$.

Here, $\{(x_3, E)\} \not\subseteq \{(x_1, E), (x_2, E), (x_3, E)\}$, showing that equality does not hold.

Definition 2.5

For any soft topological space $(X, \tilde{\tau}, E)$ and $(A, E) \subseteq X$, a point $x \in X$ is called SS^M -limit point of (A, E) if every SS^M -open set containing x contains points of (A, E) other than x . The set of all limit points of (A, E) called SS^M -derived set of (A, E) and is denoted by $SS^M d(A, E)$.

Definition 2.6

The soft simply derived of (A, E) , denoted by $SS^M d(A, E)$ is the set of all soft limit points of (A, E) . A soft limit point of (A, E) is point (x, E) such that every soft open neighborhood of (x, E) intersects (A, E) at some point other than (x, E) itself.

Theorem 2.4

For any soft topological space $(X, \tilde{\tau}, E)$ and $(A, E), (B, E) \subseteq X$, the following statements hold:

- 1- If $(A, E) \subseteq (B, E)$, then $SS^M d(A, E) \subseteq SS^M d(B, E)$.
- 2- $SS^M d((A, E) \tilde{\cup} (B, E)) \subseteq SS^M d(A, E) \tilde{\cup} SS^M d(B, E)$.
- 3- $SS^M d((A, E) \tilde{\cap} (B, E)) \subseteq SS^M d(A, E) \tilde{\cap} SS^M d(B, E)$.
- 4- (A, E) is SS^M closed set iff $SS^M d((A, E)) \subseteq (A, E)$.
- 5- $SS^M d((A, E)) = (A, E) \tilde{\cup} SS^M d((B, E))$.

Proof

1- Let $(x, E) \in SS^M d((A, E))$. This means that every soft open neighborhood of (x, E) intersects (A, E) at some point other than (x, E) . Since $(A, E) \subseteq (B, E)$, any intersection with (A, E) also implies an intersection with (B, E) . Therefore, every soft open neighborhood of (x, E) intersects (B, E) at some point other than (x, E) , so $(x, E) \in SS^M d((B, E))$.

Thus, $SS^M d((A, E)) \subseteq SS^M d((B, E))$.

2- Let $(x, E) \in SS^M d((A, E) \tilde{\cup} (B, E))$. Then (x, E) is a soft limit point of either (A, E) or (B, E) . If $(x, E) \in SS^M d((A, E))$, then every soft open neighborhood of (x, E) intersects (A, E) at some point other than (x, E) , since $(A, E) \subseteq (A, E) \tilde{\cup} (B, E)$, this implies that every soft open neighborhood of (x, E) also intersects $(A, E) \tilde{\cup} (B, E)$.

Similarly, if $(x, E) \in SS^M d((B, E))$, then every soft open neighborhood of (x, E) intersects (B, E) at some point other than (x, E) , and this intersects $(A, E) \tilde{\cup} (B, E)$. Therefore, $(x, E) \in SS^M d((A, E) \tilde{\cup} (B, E))$, and the claim holds.

3- Let $(x, E) \in SS^M d((A, E) \tilde{\cap} (B, E))$, this means that every soft open neighborhood of (x, E) intersects $(A, E) \tilde{\cap} (B, E)$ at some point other than (x, E) .

Since $(A, E) \tilde{\cap} (B, E) \subseteq (A, E)$ and $(A, E) \tilde{\cap} (B, E) \subseteq (B, E)$, every soft open neighborhood of

(x, E) also intersects (A, E) and (B, E) at some point other than (x, E) . Therefore $(x, E) \in SS^M d((A, E))$ and $(x, E) \in SS^M d((B, E))$, so $(x, E) \in SS^M d((A, E)) \cap SS^M d((B, E))$. Thus, $SS^M d((A, E) \cap (B, E)) \subseteq SS^M d((A, E)) \cap SS^M d((B, E))$.

4- $\Rightarrow$ If (A, E) is soft simply closed, then its complement is soft simply open. By definition, no point in the complement of (A, E) can be a soft limit point of (A, E) . Therefore, all soft limit points of (A, E) must lie within (A, E) , so $SS^M d((A, E)) \subseteq (A, E)$.

$\Leftarrow$ If $SS^M d((A, E)) \subseteq (A, E)$, then no point outside (A, E) is a soft limit point of (A, E) . Thus means that the complement of (A, E) is soft simply open, so (A, E) is soft simply closed.

5- $\Rightarrow$ Let $(x, E) \in D(A, E)$. By definition, (x, E) is either a point in (A, E) or a soft limit point of (A, E) . If $(x, E) \in (A, E)$, then $(x, E) \in (A, E) \cup D(B, E)$. If (x, E) is a soft limit point of (A, E) , then $(x, E) \in D(A, E)$ and this $(x, E) \in (A, E) \cup D(B, E)$.

$\Leftarrow$ let $(x, E) \in (A, E) \cup D(B, E)$. Then if $(x, E) \in (A, E)$ then $(x, E) \in D(B, E)$ because $(A, E) \subseteq D(B, E)$. If $(x, E) \in D(B, E)$, then (x, E) is a soft limit point of (B, E) since $(B, E) \subseteq (A, E)$, any soft limit point of (B, E) is also a soft limit point of (A, E) , thus $(x, E) \in D(A, E)$ and $SS^M d((A, E)) = (A, E) \cup SS^M d((B, E))$.

Definition 2.7

The soft simply derived set of a soft (A, E) , denoted as $D(A, E)$, is the set of all soft limit points of (A, E) . A soft limit point $(x, E) \in X$ is a point such that every soft open neighborhood of (x, E) intersects (A, E) at some point other than (x, E) .

Definition 2.8

For any soft topological space (X, τ, E) and $(A, E) \subseteq X$ is called SS^M -dense in X iff $SS^M cl((A, E)) = X$. The family of all SS^M -dense sets in X will be denoted by $SS^M D(X)$.

Theorem 2.5

For any soft topological space (X, τ, E) and $(U, E) \subseteq X$, the following statements are equivalent:

- 1- (U, E) is SS^M -dense in X .
- 2- The only SS^M -closed set containing (U, E) is X .
- 3- $SS^M int(X - (U, E)) = \emptyset$.

Proposition 2.1

For any soft topological space (X, τ, E) and $(U, E) \in SS^M D(X)$, the following statements hold:

- 1- $SS^M b((U, E)) = SS^M cl(X - (U, E))$.
- 2- $SS^M ext((U, E)) = \emptyset$.

Proof

Obvious.

Definition 2.9

For any soft topological space (X, τ, E) a soft set (A, E) of X is said to be

- 1- SS^M -nowhere dense if $SS^M \text{int}(SS^M \text{cl}(A, E)) = \tilde{\emptyset}$.
- 2- SS^M -residual if $SS^M \text{cl}(X - (A, E)) = X$ or $SS^M \text{int}(A, E) = \tilde{\emptyset}$.

Remark 2.4

For any soft topological space $(X, \tilde{\tau}, E)$ we have, SS^M -nowhere dense is SS^M -residual from the fact that $SS^M \text{int}((A, E)) \subseteq SS^M \text{int}(SS^M \text{cl}(A, E))$ for every $(A, E) \subseteq X$.

Proposition 2.2

For any soft topological space $(X, \tilde{\tau}, E)$, a subset $(A, E) \subseteq X$ is SS^M -nowhere dense in X if $(A, E) \subseteq SS^M \text{cl}(X - SS^M \text{cl}((A, E)))$.

Proof

Let $(B, E) = SS^M \text{cl}(A, E)$, then the condition becomes $(A, E) \subseteq SS^M \text{cl}(X - (B, E))$, we need to show that (A, E) is soft simply nowhere dense, i.e. prove that $SS^M \text{int}(B, E) = \tilde{\emptyset}$. Suppose for condition, that $SS^M \text{int}(B, E) \neq \tilde{\emptyset}$, then there exists a non-empty soft simply open set $(U, E) \subseteq (B, E) = SS^M \text{cl}(A, E)$, but $(A, E) \subseteq SS^M \text{cl}(X - (B, E))$, so (U, E) must intersect $(X - (B, E))$ since (U, E) is open and $(B, E) = \text{cl}(A, E)$, this contradicts $(U, E) \subseteq (B, E)$. Hence, $SS^M \text{int}(B, E) = \tilde{\emptyset}$.

Definition 2.10

Let $(X, \tilde{\tau}, E)$ be a soft topological space, and let (A, E) be a soft subset of X ,

- 1- The soft boundary of (A, E) denoted as: $Sb(A, E) = Scl(A, E) \tilde{\cap} Scl(X - (A, E))$.
- 2- A soft set (B, E) is called soft simply residual if either:
 $SS^M \text{cl}(X - (B, E)) = X$ or $SS^M \text{int}((B, E)) = \tilde{\emptyset}$.

Theorem 2.6

Let (A, E) be a soft simply open set in a soft topological space $(X, \tilde{\tau}, E)$. Then the soft simply boundary of (A, E) , is soft simply nowhere dense.

Proof

Since (A, E) is soft simply open, we can write $(A, E) = (G, E) \tilde{\cup} (F, E)$ where (G, E) is soft open and (F, E) is soft nowhere dense. Analyze the closure : $SS^M \text{cl}(A, E) = SS^M \text{cl}(G, E) \tilde{\cup} SS^M \text{cl}(F, E) = SS^M \text{cl}(G, E)$, since (F, E) is nowhere dense. Analyze the boundary : $SS^M b(A, E) = SS^M \text{cl}(A, E) \tilde{\cap} SS^M \text{cl}(A^c, E) = SS^M \text{cl}(G, E) \tilde{\cap} SS^M \text{cl}((G \tilde{\cup} F)^c, E) = SS^M \text{cl}(G, E) \tilde{\cap} SS^M \text{cl}(G^c \tilde{\cap} F^c, E) \subseteq SS^M \text{cl}(G, E) \tilde{\cap} SS^M \text{cl}(G^c, E) = SS^M b(G, E)$ it is the ordinary soft boundary of (G, E) . Now we prove that (A, E) is nowhere dense, let the soft interior of the closure of the boundary : $SS^M \text{int}(SS^M \text{cl}(SS^M b(A, E))) \subseteq SS^M \text{int}(SS^M \text{cl}(SS^M b(G, E)))$, but $SS^M b(G, E)$ is soft nowhere dense in ordinary soft topology, so $SS^M \text{int}(SS^M \text{cl}(SS^M b(G, E))) = \tilde{\emptyset}$. Therefore $SS^M \text{int}(SS^M \text{cl}(SS^M b(A, E))) = \tilde{\emptyset}$.

Theorem 2.7

For any soft topological space $(X, \tilde{\tau}, E)$ and $(A, E) \cong X$, the sets $(A, E) \tilde{\cap} SS^M cl(X - (A, E))$ and $SS^M cl(A, E) \tilde{\cap} (X - (A, E))$ are SS^M -residual.

Proof

We need to prove that $(A, E) \tilde{\cap} SS^M cl(X - (A, E))$ is soft simply residual, and prove $SS^M cl(A, E) \tilde{\cap} (X - (A, E))$ is soft simply residual. Now, let $S_1 = (A, E) \tilde{\cap} SS^M cl(X - (A, E))$, by definition $SS^M cl(X - (A, E))$ is the smallest soft closed set containing $(X - (A, E))$. If $SS^M cl(X - (A, E)) = X$, then $S_1 = (A, E) \tilde{\cap} X = (A, E)$, so that (A, E) is soft simply residual because $SS^M cl(X - (A, E)) = X$. If $SS^M cl(X - (A, E)) \neq X$, then $S_1 = (A, E) \tilde{\cap} SS^M cl(X - (A, E))$, and the soft simply interior of $S_1 \neq \tilde{\emptyset}$ because S_1 lies entirely within $SS^M cl(X - (A, E))$, which has no soft interior points in (A, E) . Thus S_1 is soft simply residual by definition. Also, let $S_2 = SS^M cl(A, E) \tilde{\cap} (X - (A, E))$, by definition $SS^M cl(A, E)$ is the smallest soft closed set containing (A, E) , the set $(X - (A, E))$ is the complement of (A, E) , which is soft open if (A, E) is soft closed, or soft closed if (A, E) is soft open. If $SS^M cl(A, E) = X$, then, $S_2 = X \tilde{\cap} (X - (A, E)) = (X - (A, E))$, so that $(X - (A, E))$ is soft simply residual because $SS^M cl(A, E) = X$. If $SS^M cl(A, E) \neq X$, then $S_2 = SS^M cl(A, E) \tilde{\cap} (X - (A, E))$, and the soft simply interior of S_2 is $\tilde{\emptyset}$ because S_2 lies entirely within $X - (A, E)$, which has no soft interior points in $SS^M cl(A, E)$. Thus, S_2 is soft simply residual by definition.

Theorem 2.8

The boundary for any soft set contains the union of soft two SS^M -residual sets.

Proof

We aim to prove that the soft boundary of (A, E) contains the union of two soft simply residual sets then we can define the first soft simply residual set $S_1 = (A, E) \tilde{\cap} SS^M cl(X - (A, E))$, define the second soft simply residual set $S_2 = SS^M cl(A, E) \tilde{\cap} (X - (A, E))$. Prove that S_1 and S_2 are soft simply residual:
For S_1 : if $SS^M cl(X - (A, E)) = X$, then $S_1 = (A, E) \tilde{\cap} X = (A, E)$. Which is soft simply residual by definition. If $SS^M cl(X - (A, E)) \neq X$, then the soft simply interior of S_1 is $\tilde{\emptyset}$, making S_1 soft simply residual.

For S_2 : if $SS^M cl((A, E)) = X$, then $S_2 = X \tilde{\cap} (X - (A, E)) = (X - (A, E))$ which is soft simply residual. If $SS^M cl((A, E)) \neq X$, then the soft simply interior of S_2 is $\tilde{\emptyset}$, making S_2 soft simply residual, so that $S_1 \tilde{\cup} S_2 = [(A, E) \tilde{\cap} S]$ contained in the soft boundary of (A, E) because $S_1 \cong Scl(X - (A, E))$, and $S_1 \cong (A, E)$, so $S_1 \cong Sb(A, E)$, $S_2 \cong Scl((A, E))$ and $S_2 \cong (X - (A, E))$, so Therefore $S_1 \tilde{\cup} S_2 \cong Sb(A, E)$.

CONCLUSION

The study has systematically explored the concept of soft simply open sets within the framework of soft topological spaces, establishing their foundational properties and applications. The most important results:

1- Theoretical Contributions:

- i- We have proven that soft simply open sets are more accurate to study the separation axioms (e.g., near normality), compared to traditional methods.
- ii- We have proven the relationship between the soft simply open (closed) sets and their soft interior (closure) operations.

2- Practical Implications:

- i- These applications have been used in uncertain data modeling and AI-based classification, demonstrating their importance in dealing with fuzzy, inaccurate information.

2- In this study, we simplify the evidence in soft topology while maintaining robustness, as demonstrated in our comparative analysis

3- Future Directionsi- Extending this work to soft compactness could further enrich the theory

- ii- Finally, soft simply open sets bridge theoretical topology with applied mathematics, paving the way for innovative and rapid solutions in data science.

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