Al-Mukhtar Journal of Basic Sciences 22 (3): 221-232, 2024
Doi: https://doi.org/10.54172/dg3v6n81

**Research Article** **Open Access**

# Existence of Local Solutions for A Chemotaxis Navier Stokes System Modeling Cellular Swimming in Fluid Drops with Logistic Source

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**Received:**
12 October 2024

**Accepted:**
04 December 2024

**Publish online:**
31 December 2024

**Abstract**

In this paper, we are concerned with the Cauchy problem for the three-dimensional chemotaxis system with an indirect signal production mechanism involving a diffusive partial differential equation. Which describes the motion of bacteria, Eukaryotes, in a fluid. Precisely, for the Chemotaxis-Navier–Stokes system modeling cellular swimming in fluid drops. We established the existence of local solutions to the compressible chemotaxis equation. We proved the local existence of the Cauchy problem (1.1)-(1.2) in $\mathbb{R}^3$ with the small initial data by using the energy method.

**Keywords:** Chemotaxis system, Energy method, nonlinear diffusion.

## INTRODUCTION

Chemotaxis refers to the directional movement of organisms in response to certain chemicals in their environments, which plays an essential role in various biological processes such as wound healing, cancer invasion, and avoidance of predators (Di Francesco et al., 2010). It has attracted considerable attention due to its critical role in a wide range of biological phenomena. In 1970, (Keller & Segel, 1970) derived the following chemotaxis model

$$n_t = \nabla.(D(n)\nabla n) - \nabla.(nS(n)\nabla c), \quad x \in \Omega, t > 0,$$
$$c_t = \Delta c - c + n, \qquad x \in \Omega, t > 0. \qquad (1.1)$$

Where $n(x,t) and\ c(x,t)$ represent the density of the bacteria and oxygen concentration at position $x$ and $t > 0$, respectively. The function $S(n)$ measures the chemotactic sensitivity and $D(n)$ is the diffusion function. There are a large number of results about whether the solutions for the Neumann boundary problem of $(1.1)$ globally exist or blow up in finite time. One can refer to (Horstmann & Winkler, 2005; Winkler, 2016; Zhang & Li, 2015) to find more related results. If we consider the framework in which the chemical is produced by the cells indirectly, the corresponding chemotaxis model becomes the following Keller-Segel system with indirect signal production:

$$n_t = \nabla.(D(n)\nabla n) - \nabla.(nS(n)\nabla c), \quad x \in \Omega, t > 0,$$

$$v_t = \Delta v - v - w, \qquad x \in \Omega, t > 0,$$

$$w_t = \Delta w - w - v, \qquad x \in \Omega, t > 0. \quad (1.2)$$

In a bounded domain $\Omega \in \mathbb{R}^3$ with smooth boundary, where the variables $n$, $v$, and $w$ represent the density of cells, the concentration of signal, and the concentration of the chemical, respectively. If $N \leq 3, D(n) \equiv 1$ and $S(n) = \chi$ with $\chi > 0$, (Fujie & Senba, 2017) proved that the homogeneous Neumann boundary problem of the system (1.2) possesses a unique and globally bounded classical solution. More recent observations show that in certain cases of chemical movement in liquid environments, the interaction between cells and liquids may be significant (see e.g. (Chae et al., 2014; Cieślak & Stinner, 2012; Shi et al., 2017) and references therein). It is also important to consider the biological situation populations of bacteria may reproduce according to a logistical plan. It can be observed experimentally that spatial patterns may arise spontaneously from initially almost homogeneous distributions of bacteria (Dombrowski et al., 2004). When bacteria of the species Bacillus subtilis are suspended in the fluid (Tuval et al., 2005). conducted a detailed experimental and theoretical study on the interaction of bacterial chemotaxis, chemical diffusion, and fluid convection. In particular, by placing a water droplet containing Bacillus subtilis in a chamber with its upper surface open to the atmosphere, they observed that bacterial cells quickly get densely packed in a relatively thin liquid layer below the water-air interface through which the oxygen diffuses into the water droplet. For such processes, a mathematical model was proposed in (Wang et al., 2018; Winkler, 2012), where it is assumed that the main responsible mechanisms are the chemical movement of bacteria towards the oxygen they consume and the effect of gravity on the movement of the fluid by heavier bacteria, and the thermal transport of both cells and oxygen through the fluid see also (Chae et al., 2012). However, in different situations, bacterial migration is greatly influenced by changes in their environment (Cieślak & Laurençot, 2010). If cells consume the chemical signal, (Tuval et al., 2005) explored the following chemotactic Navier-Stokes system:

$$n_t + u.\nabla n = \nabla.(D(n)\nabla n) - \nabla.(nS(x,n,c)\nabla c), \quad x \in \Omega, t > 0,$$

$$c_t + u.\nabla c = \Delta c - nf(c), \qquad x \in \Omega, t > 0,$$

$$u_t + \kappa(u.\nabla u)u + \nabla p = \Delta u + n\nabla\varphi, \qquad x \in \Omega, t > 0,$$

$$\nabla.u = 0\,, \qquad x \in \Omega, t > 0 \qquad (1.3)$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary, where $f(c)$ measures the rate at which cells consume oxygen, and $S(x,n,c)$denotes a tensor-valued (or scalar) chemotactic sensitivity. Here $u, p, \varphi,$ and $\kappa \in$ R denote the velocity field, the associated pressure of the fluid, the potential of the gravitational field, and the strength of nonlinear fluid convection, respectively. By the chemical consumption setting and the maximum principle of the parabolic equations, one can directly deduce that $c$ is uniformly bounded from the second equation of (1.3), which leads to it being more intensively studied than the framework with signal production by the cells.(Lorz, 2010) discussed the local existence of weak solutions to in a bounded domain in $R^d$, d = 2,3. In the case of homogeneous boundary conditions of Neumann type of $n$ and $c$, and of Dirichlet type for $u$, (Winkler, 2014) showed that the global weak solutions to (1.3). For more literature about this system, one can refer to (Hillen & Painter, 2009; Winkler, 2017a) and the references therein for details. There are also more results about chemotaxis systems with nonlinear chemotaxis sensitivity functions, for which we refer to (Bellomo et al., 2015; Cieślak & Winkler, 2008, 2017; Hou & Wang, 2019; Li, 2019; Pan & Wang, 2021; Rosen, 1978; Tao & Winkler, 2012; Winkler, 2017b). (Hattori & Lagha, 2021a, 2021b) showed the global existence and asymptotic behavior of the solutions for a chemotaxis system with chemoattractant and repellent in three dimensions. In this paper, we are concerned with the following initial value problem of the Keller-Segel-Nevier-Stokes system with nonlinear diffusion:

$$\partial_t n - \Delta n + u.\nabla n = -\chi \nabla.(\mathrm{n}\nabla c) + n(n - n_\infty)$$

$$\partial_t u - \gamma \Delta u + u.\nabla u + \nabla \pi = -n\nabla\varphi \qquad (1.4)$$

$$\partial_t c - \Delta c + u.\nabla c = -nc$$

$$\nabla.u = 0 \quad t > 0, \qquad x \in \mathbb{R}^3.$$

With initial data

$$(n,u,c)|_{t=0} = \big(n_0(x), u_0(x), c_0(x)\big),\ x \in \mathbf{R}^3, \qquad (1.5)$$

where$\big(n_0(x), u_0(x), c_0(x)\big) \to (n_\infty, 0,0)$ as $|\mathrm{x}| \to \infty$. Here $n = n(t,x), c = c(t,x), and\ u = u(t,x)$ denote the bacterial concentration, the oxygen concentration, and the fluid velocity field respectively. In addition, $\pi = \pi(t,x)$ is unknown pressure and $\varphi$ is the gravitational potential function. The term $-\chi\nabla.(\mathrm{n}\nabla c)$ reflects the attractive movement of cells. While $n_0 = n_0(x)$, $u_0 = u_0(x)$, and $c_0 = c_0(x)$, are the given functions, the constants $\gamma > 0$, $\chi > 0$. Where $n_\infty$ is a non-negative constant. A simple model case can be obtained upon the choices $\nabla\varphi = const.$, $\chi = 1$.

This paper is organized as follows, the first section is this introduction, which describes of some of the models used in chemotaxis, their rationale, and a very brief summary of the results obtained. In Section 2, we present notations and some assumptions that will be heavily used throughout the whole paper and state our main result. In Section 3, we prove the local-in-time existence of a regular solution for three-dimensional chemotaxis system with incompressible Navier-Stokes equations.

**2. Main result**

We first introduce some notations that we will use later in this paper. For $1 \le P \le \infty$, we denote $L^p$ for the Lebesgue space on $\Omega$, and the norms in the space $L^p(\boldsymbol{R}^3)$ are denoted by $\|.\|_{\mathrm{p}}$. For any integer $N \ge 0$, we use $\mathrm{H^N}$ to denote the Sobolev space $\mathrm{H^N}(\boldsymbol{R}^3)$. Set $\mathrm{L}^2 = \mathrm{H}^0$, the norm of $\mathrm{H^N}$ is denoted by $\|.\|_{\mathrm{H^N}}$. We set $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ for a multi-index $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and length of $\alpha$ is $|.| = \alpha_3 + \alpha_2 + \alpha_3$. C and $C_i$, where $i = 1,2$, denote some positive (generally small) constant, where both C and $C_i$ may take different values in different places. Let us denote the space

$$X(0,T) = \{n - n_\infty, u, c \in \mathrm{C}\big([0,\mathrm{T}]; \mathrm{H}^3(\mathbf{R}^3)\big) \cap \mathrm{C}^1\big([0,\mathrm{T}]; \mathrm{H}^1(\mathbf{R}^3)\big), \nabla(n - n_\infty), \nabla u, \nabla\, c \in \mathrm{L}^2\big([0,\mathrm{T}]; \mathrm{H}^3(\mathbf{R}^3)\big)\}.$$

The main goal of this paper is to establish the existence of unique local solutions in three dimensions around a constant state $(n_\infty, 0, 0)$ for the above system (1.4). The main result of this paper is stated as follows:

**Theorem 2.1**. There exists a positive number $\varepsilon_0$ such that if

$$\|n_0, u_0, c_0\|_{\mathrm{H}^3} \le \varepsilon_0,$$

then the Cauchy problems (1.4)-(1.5) of the Keller-Segel-Nevier-Stokes system admits a unique local solution $(n,u,c)$ with

$$(n - n_\infty, u, c) \in X(0,T).$$

The proof of the existence of local solutions in Theorem 2.1 by constructing a sequence of approximation functions based on iteration and some basic energy estimates. Let $U(t) = (n, u, c)$ be a smooth solution to the Cauchy problem of the Chemotaxis system (1.4) with initial data $U_0 = (n_0, u_0, c_0)$.

We set:

$$n(t,x) = n_\infty + \sigma(t,x).$$

Then, the Cauchy problem (1.4) and (1.5) are reformulated as

$$\begin{aligned}
&\partial_t\sigma - \Delta\sigma + u.\nabla\sigma + n_\infty\sigma = -\nabla.(\sigma\nabla c) - n_\infty\Delta c + \nabla\sigma^2\\
&\partial_t u - \gamma\Delta u + u.\nabla u = -\nabla\pi - (\sigma + n_\infty)\nabla\varphi\\
&\partial_t c + -\Delta c + u.\nabla c = -(\sigma + n_\infty)c\\
&\nabla.u = 0 \quad t > 0, x \in \mathbb{R}^3, \qquad (2.1)
\end{aligned}$$

with initial data

$$(\sigma, u, c)|_{t=0} = (\sigma_0, u_0, c_0) \to (0,0,0) \text{ as } |x| \to \infty, \quad (2.2)$$

where $\sigma_0 = n_0 - n_\infty$

## 3. Existence of local solutions

This section is devoted to the proof of Theorem 2.1. We construct the sequence $(n^j, u^j, c^j)_{j\geq 0}$ by solving iteratively the Cauchy problems on the following linear equations

$$\begin{aligned}
&\partial_t n^{j+1} + u^j.\nabla n^{j+1} = \Delta n^{j+1} - \nabla.\left(n^j\nabla c^{j+1}\right) + n^j(n^j - n_\infty)\\
&\partial_t u^{j+1} + u^j.\nabla u^{j+1} = \gamma\Delta u^{j+1} - \nabla\pi^{j+1} - n^j\nabla\varphi\\
&\partial_t c^{j+1} + u^j.\nabla c^{j+1} + \Delta c^{j+1} = -n^j c^{j+1} \qquad (3.1)\\
&\nabla.u^{j+1} = 0 \quad t > 0, \quad x \in \mathbb{R}^3,
\end{aligned}$$

with initial data

$$\left(n^{j+1}, u^{j+1}, c^{j+1}\right)\Big|_{t=0} = (n_0, u_0, c_0), \quad x \in \mathbb{R}^3, \ (3.2)$$

for $j \geq 0$, where $(n^0, c^0, u^0)|_{t=0} = (n_\infty, 0,0)$ is set at initial step. Now, we set $n^j = \sigma^j + n_\infty$, then (3.1)-(3.2) can be rewritten as

$$\begin{cases}
\partial_t\sigma^{j+1} + u^j.\nabla\sigma^{j+1} + n_\infty\sigma^{j+1} = \Delta\sigma^{j+1} - \nabla.\left(\sigma^j\nabla c^{j+1}\right) - n_\infty\Delta c^{j+1} + \nabla\sigma^{j2}\\
\partial_t c^{j+1} + u^j.\nabla c^{j+1} + \Delta c^{j+1} = -\left(\sigma^j + n_\infty\right)c^{j+1}\\
\partial_t u^{j+1} + u^j.\nabla u^{j+1} = \gamma\Delta u^{j+1} - \nabla\pi^{j+1} - \left(\sigma^j + n_\infty\right)\nabla\varphi\\
\nabla.u^{j+1} = 0 \quad t > 0, \quad x \in \mathbb{R}^3, \qquad (3.3)
\end{cases}$$

with initial data

$$(\sigma^{j+1}, c^{j+1}, u^{j+1})|_{t=0} = (\sigma_0, c_0, u_0) \to (0,0,0) \qquad (3.4)$$

as $|x| \to \infty$, for $j \geq 0$. In what follows, let us write $A^j = \left(\sigma^{j+1}, u^{j+1}, c^{j+1}\right)_{j\geq 0}$ and $A_0 = (\sigma_0, u_0, c_0)$, where $A^0 \equiv (0,0,0)$. Next, we prove that $(A^j)_{j\geq 0}$ is a Cauchy sequence in the Banach space $C([0,T_1];H^3)$ for $T_1 > 0$ suitable small. At last, by taking the limit and continuous argument, we prove that $(\sigma, u, c)$ is a local solution to (2.1)-(2.2). Now, we can state the following result:

**Theorem 3.1**.

Suppose $\|A_0\|_{H^3} \leq \varepsilon_1$, for small constants $\varepsilon_1 > 0$, $T_1 > 0$, $B_1 > 0$. Then for each $j \geq 0$, $A^j \in C([0, T_1)];\ H^3)$, is well defined and

$$\sup_{0\leq t\leq T_1} \left\|A^j(t)\right\|_{H^N} \leq B_1, \qquad (3.5)$$

Moreover, $(A^j)_{j\geq 0}$ is a Cauchy sequence in Banach space $C([0, T_1];\ H^3)$, the corresponding limit function denoted by A belongs to $C([0, T_1]; H^3)$ with

$$\sup_{0\leq t\leq T_1} \|A(t)\|_{H^N} \leq B, \quad (3.6)$$

and $A = (\sigma, u, c)$ is a solution to the Cauchy problem (2.1)-(2.2) over $[0, T_1]$. The Cauchy problem (2.1)-(2.2) admits at most one solution $A \in C([0, T_1]; H^3)$, which satisfies (3.6).

**proof .**

We begin by focusing our attention on the proof of (3.5), which will be given by an inductive argument. The trivial case is $j = 0$ since $A^0 = (0,0,0)$ by the assumption at initial step. Suppose that (3.5) holds for some $j \geq 0$ where is small enough. To prove (3.5) holds for j+1, we need some energy estimates on $(\sigma^{j+1}, c^{j+1}, u^{j+1})$.

Applying $\partial^\alpha$ to both sides of the first equation of (3.3), multiplying by $\partial^\alpha\sigma^{j+1}$, and integrating over $\mathbb{R}^3$ with $|\alpha| \leq 3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\partial^\alpha\sigma^{j+1}\right|^2 dx + \int_{\mathbb{R}^3}\partial^\alpha\Delta\sigma^{j+1}\,\partial^\alpha\sigma^{j+1}dx + n_\infty\int_{\mathbb{R}^3}\left|\partial^\alpha\sigma^{j+1}\right|^2 dx,$$

$$= -\int_{\mathbb{R}^3}\partial^\alpha\left(u^j.\nabla\sigma^{j+1}\right)\partial^\alpha\sigma^{j+1}\,dx - \int_{\mathbb{R}^3}\partial^\alpha\nabla.\left(\sigma^j\nabla c^{j+1}\right)\partial^\alpha\sigma^{j+1}\,dx$$

$$+n_\infty\int_{\mathbb{R}^3}\partial^\alpha\Delta c^{j+1}\,\partial^\alpha\sigma^{j+1}dx + \int_{\mathbb{R}^3}\partial^\alpha\sigma^{j+1}\,\partial^\alpha\sigma^{j2}\ dx.$$

By using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\partial^\alpha\sigma^{j+1}\right|^2 dx + \int_{\mathbb{R}^3}\left|\partial^\alpha\nabla\sigma^{j+1}\right|^2 dx + n_\infty\int_{\mathbb{R}^3}\left|\partial^\alpha\sigma^{j+1}\right|^2 dx,$$

$$= -\int_{\mathbb{R}^3}\partial^\alpha\left(u^j.\nabla\sigma^{j+1}\right)\partial^\alpha\sigma^{j+1}\,dx - \int_{\mathbb{R}^3}\partial^\alpha\left(\sigma^j\nabla c^{j+1}\right)\partial^\alpha\nabla\sigma^{j+1}\,dx$$

$$+n_\infty\int_{\mathbb{R}^3}\partial^\alpha\nabla c^{j+1}\,\partial^\alpha\nabla\sigma^{j+1}dx + \int_{\mathbb{R}^3}\partial^\alpha\sigma^{j+1}\,\partial^\alpha\sigma^{j2}\ dx. \quad (3.7)$$

By using the Cauchy inequality the terms on the right-hand side are bounded by

$$C\|u^j\|_{H^3}\|\nabla\sigma^{j+1}\|_{H^3}^2 + C\|\sigma^j\|_{H^3}\|\nabla c^{j+1}\|_{H^3}\|\nabla\sigma^{j+1}\|_{H^3}$$
$$+n_\infty\left(\|\nabla c^{j+1}\|_{H^3}\|\nabla\sigma^{j+1}\|_{H^3}\right) + C\|\sigma^j\|_{H^3}\|\nabla\sigma^{j+1}\|_{H^3} + \|\sigma^j\|_{H^3}^2. \quad (3.8)$$

Then, by taking the summation over $|\alpha| \le 3$, we have

$$\frac{1}{2}\frac{d}{dt}\|\sigma^{j+1}\|_{H^3}^2 + \frac{1}{2}\|\nabla\sigma^{j+1}\|_{H^3}^2 + n_\infty\|\sigma^{j+1}\|_{H^3}^2$$
$$\le C\|\nabla c^{j+1}\|_{H^3}^2 + C\|(u^j,\sigma^j)\|_{H^3}^2\|\nabla(\sigma^{j+1},c^{j+1})\|_{H^3}^2 + C\|\sigma^j\|_{H^3}^2. \quad (3.9)$$

By the same way, for $(3.3)_2$ on $c^{j+1}$, one has

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial^\alpha c^{j+1}|^2dx + \int_{\mathbb{R}^3}|\nabla\partial^\alpha c^{j+1}|^2dx + n_\infty\int_{\mathbb{R}^3}|\partial^\alpha c^{j+1}|^2dx$$
$$= -\int_{\mathbb{R}^3}\partial^\alpha(u^j.\nabla c^{j+1})\,\partial^\alpha c^{j+1}dx - \int_{\mathbb{R}^3}\partial^\alpha(\sigma^j c^{j+1})\,\partial^\alpha c^{j+1}dx\ .$$

The terms on the right-hand side of the previous inequality are bounded by

$$c\|u^j\|_{H^3}\|\nabla c^{j+1}\|_{H^3}^2 + c\|\sigma^j\|_{H^3}\|\nabla c^{j+1}\|_{H^3}\|c^{j+1}\|_{H^3}.$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\|c^{j+1}\|_{H^3}^2 + \frac{1}{2}\|\nabla c^{j+1}\|_{H^3}^2 + \frac{n_\infty}{2}\|c^{j+1}\|_{H^3}^2 \le C\|(u^j,\sigma^j)\|_{H^3}^2\|\nabla c^{j+1}\|_{H^3}^2. \quad (3.10)$$

Similarly, for the $(3.3)_3$ on $u^{j+1}$, we get that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial^\alpha u^{j+1}|^2dx + \gamma\int_{\mathbb{R}^3}|\nabla\partial^\alpha u^{j+1}|^2dx =$$
$$-\int_{\mathbb{R}^3}\partial^\alpha(u^j.\nabla u^{j+1}).\partial^\alpha u^{j+1}dx + \int_{\mathbb{R}^3}\partial^\alpha(\nabla\sigma^j\varphi).\partial^\alpha u^{j+1}dx.$$

Where the right-hand side of the previous equation is bounded by

$$C\|u^j\|_{H^3}\|\nabla u^{j+1}\|_{H^3}^2 + C\|\sigma^j\|_{H^3}^2 + \frac{1}{2}\|\nabla u^{j+1}\|_{H^3}^2.$$

Then, after taking summation over $|\alpha| \le 3$ and using the Cauchy inequality, one has

$$\frac{1}{2}\frac{d}{dt}\|u^{j+1}\|_{H^3}^2 + \frac{\gamma}{2}\|\nabla u^{j+1}\|_{H^3}^2 \le C\|\sigma^j\|_{H^3}^2 + C\|u^j\|_{H^3}^2\|\nabla u^{j+1}\|_{H^3}^2. \quad (3.11)$$

Then the linear combination (3.9)+(3.10)× d+(3.11) leads to

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma^{j+1}\|_{H^3}^2 + d\|c^{j+1}\|_{H^3}^2 + \|u^{j+1}\|_{H^3}^2\right)$$

$$+C_1\left\|\nabla(\sigma^{j+1}, c^{j+1}, u^{j+1})\right\|_{H^3}^2 + C_2\left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2$$

$$\leq C\left\|\sigma^j\right\|_{H^3}^2 + C\left\|\sigma^j, c^j, u^j\right\|_{H^3}^2\left\|\nabla(\sigma^{j+1}, c^{j+1}, u^{j+1})\right\|_{H^3}^2, \qquad (3.12)$$

By choosing $d > 0$ large enough. Further, after Integrating (3.12) over $[0, t]$ for all $t \in [0, T_1]$, we have

$$\left\|A^{j+1}(t)\right\|_{H^3}^2 + C_1\int_0^t \left\|\nabla A^{j+1}(s)\right\|_{H^3}^2 ds + C_2\int_0^t \left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2$$

$$\leq C\left\|A_0\right\|_{H^3}^2 + C\int_0^t \left\|A^j(s)\right\|_{H^3}^2 ds + C\int_0^t \left\|A^j(s)\right\|_{H^3}^2 \left\|\nabla A^{j+1}(s)\right\|_{H^3}^2 ds, \quad (3.13)$$

for some positive constants $C_1$ *and* $C_2$. From the inductive assumption, the previous inequality can be re-estimated as

$$\left\|A^{j+1}(t)\right\|_{H^3}^2 + C_1\int_0^t \left\|\nabla A^{j+1}(s)\right\|_{H^3}^2 ds + C_2\int_0^t \left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2 ds$$

$$\leq C\varepsilon_1^2 + CB_1^2 T_1 + CB_1^2\int_0^t \left\|\nabla A^{j+1}(s)\right\|_{H^3}^2 ds, \qquad (3.14)$$

for any $0 \leq t \leq T_1$. Now, we take the small constants $\varepsilon_1 > 0$, $B_1 > 0$ and $T_1 > 0$. Then, we get

$$\left\|A^{j+1}(t)\right\|_{H^3}^2 + C_1\left\|\nabla A^{j+1}(t)\right\|_{H^3}^2 + C_2\left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2 \leq B_1^2, \qquad (3.15)$$

for any $0 \leq t \leq T_1$. This implies that (3.5) holds for $j+1$, Hence (3.5) is proved for all $j \geq 0$.

Next, we define

$$E(A^{j+1}(t)) := \left\|\sigma^{j+1}\right\|_{H^3}^2 + d\left\|c^{j+1}\right\|_{H^3}^2 + \left\|u^{j+1}\right\|_{H^3}^2,$$

Where the constant $d > 0$ is given in (3.12). Similar to prove (3.12), we have

$$\left|E(A^{j+1}(t)) - E(A^{j+1}(s))\right| = \left|\int_s^t \frac{d}{d\tau} E(A^{j+1}(\tau))\, d\tau\right|$$

$$\leq C\int_s^t \left\|A^j(\tau)\right\|_{H^3}^2 d\tau + C\int_s^t \left(1 + \left\|A^j(\tau)\right\|_{H^3}^2\right)\left\|\nabla A^{j+1}(\tau)\right\|_{H^3}^2 d\tau + C_2\int_0^t \left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2 d\tau,$$

$$\leq CB_1{}^2(t-s) + C\left(1 + B_1{}^2\right)\int_s^t \left\|\nabla A^{j+1}(\tau)\right\|_{H^3}^2 d\tau + C_2\int_0^t \left\|\sigma^{j+1}, c^{j+1}\right\|_{H^3}^2 d\tau, \quad (3.16)$$

for any $0 \leq s \leq t \leq T_1$. Here, The time integral on the right-hand side from the above inequality is bounded by (3.15), and hence $E(A^{j+1}(t))$ is Continuous in $t$ for each $j \geq 0$. By the same argument, we can infer that both $\left\|c^{j+1}\right\|_{H^3}^2$, and $\left\|u^{j+1}\right\|_{H^3}^2$ are continuous in $t$. From the continuity of $E\,(A^{j+1}(t))$, we can also infer the continuity of $\left\|\sigma^{j+1}\right\|_{H^3}^2$. Therefore, $\left\|A^{j+1}(t)\right\|_{H^3}^2$ is continuous in time for any $j \geq 1$.

For this step, we prove that the sequence $(A^j)_{j\geq 0}$ is a Cauchy sequence in the Banach space $C([0,T_1];H^3)$, which converges to the solution $U=(\sigma,u,c)$ of the Cauchy problem (2.1)-(2.2), and satisfies (3.6). Let us take the difference of (3.3) for $j+1$ and $j$ so that it gives

$$\partial_t(\sigma^{j+1}-\sigma^j)-\Delta(\sigma^{j+1}-\sigma^j)+n_\infty(\sigma^{j+1}-\sigma^j)=-u^j.\nabla(\sigma^{j+1}-\sigma^j)-(u^j-u^{j-1}).\nabla\sigma^j$$
$$-\nabla.\left(\sigma^j\nabla(c^{j+1}-c^j)\right)-\nabla.((\sigma^j-\sigma^{j-1})\nabla c^j)\ -n_\infty[\Delta(c^{j+1}-c^j)]\ -(\sigma^{(j+1)2}-\sigma^{j2})$$
$$\partial_t(c^{j+1}-c^j)-\Delta(c^{j+1}-c^j)+n_\infty(c^{j+1}-c^j)$$
$$=-u^j.\nabla(c^{j+1}-c^j)-(u^j-u^{j-1}).\nabla c^j-\sigma^j(c^{j+1}-c^j)-(\sigma^j-\sigma^{j-1})c^j,$$
$$\partial_t(u^{j+1}-u^j)-\gamma\Delta(u^{j+1}-u^j)\ =\ -\nabla(\pi^{j+1}-\pi^j)-u^j.\nabla(u^{j+1}-u^j)-(u^j-u^{j-1}).\nabla u^j$$
$$\nabla.(u^{j+1}-u^j)=0,\qquad t>0, x\in\mathbb{R}^3\,.$$

By using the same energy estimates as before, we have

$$\frac{1}{2}\frac{d}{dt}\left\|(\sigma^{j+1}-\sigma^j)\right\|_{H^3}^2+\left\|\nabla(\sigma^{j+1}-\sigma^j)\right\|_{H^3}^2+n_\infty\left\|(\sigma^{j+1}-\sigma^j)\right\|_{H^3}^2$$
$$\leq C\left\|(u^j-u^{j-1},\sigma^j-\sigma^{j-1})\right\|_{H^3}^2\left\|\nabla(\sigma^j,c^j)\right\|_{H^3}^2$$
$$+C\left\|(\sigma^j,u^j)\right\|_{H^3}^2\left\|\nabla(\sigma^{j+1}-\sigma^j,c^{j+1}-c^j)\right\|_{H^3}^2+n_\infty\left\|\nabla(c^{j+1}-c^j)\right\|_{H^3}^2 \qquad (3.17)$$

$$\frac{1}{2}\frac{d}{dt}\left\|(c^{j+1}-c^j)\right\|_{H^3}^2+\frac{n_\infty}{2}\left\|(c^{j+1}-c^j)\right\|_{H^3}^2+\frac{1}{2}\left\|\nabla(c^{j+1}-c^j)\right\|_{H^3}^2\leq$$
$$C\left\|(\sigma^j,u^j)\right\|_{H^3}^2\ \left\|\nabla(c^{j+1}-c^j)\right\|_{H^3}^2+C\left\|\nabla(\sigma^j-\sigma^{j-1},u^j-u^{j-1})\right\|_{H^3}^2\ \left\|\nabla c^j\right\|_{H^3}^2 \qquad (3.18)$$

$$\frac{1}{2}\frac{d}{dt}\left\|(u^{j+1}-u^j)\right\|_{H^3}^2+\frac{\gamma}{2}\left\|\nabla(u^{j+1}-u^j)\right\|_{H^3}^2$$
$$\leq C\left\|u^j\right\|_{H^3}^2\left\|\nabla(u^{j+1}-u^j)\right\|_{H^3}^2+C\left\|(u^j-u^{j-1})\right\|_{H^3}^2\ \left\|\nabla u^j\right\|_{H^3}^2$$
$$+C\left\|\sigma^j-\sigma^{j-1}\right\|_{H^3}^2. \qquad (3.19)$$

We combine the equations (3.17)-(3.19) to obtain

$$\frac{1}{2}\frac{d}{dt}E(A^{j+1}-A^j)+\frac{1}{4}\left\|\nabla(A^{j+1}-A^j)\right\|_{H^3}^2+\frac{n_\infty}{2}\ \left\|(c^{j+1}-c^j)\right\|_{H^3}^2+n_\infty\left\|(\sigma^{j+1}-\sigma^j)\right\|_{H^3}^2$$
$$\leq C\left\|A^j-A^{j-1}\right\|_{H^3}^2+C\left\|A^j\right\|_{H^3}^2\left\|\nabla(A^{j+1}-A^j)\right\|_{H^3}^2+C\left\|A^j-A^{j-1}\right\|_{H^3}^2\ \left\|\nabla A^j\right\|_{H^3}^2. \qquad (3.20)$$

By integrating over $[0,t]$ for any $0\leq t\leq T_1$ from (3.20), we obtain

$$\left\|(A^{j+1}(t)-A^j(t)\right\|_{H^3}^2+C_1\int_0^t\left\|\nabla(A^{j+1}(s)-A^j(s))\right\|_{H^3}^2ds+\frac{n_\infty}{2}\int_0^t\left\|(c^{j+1}-c^j)\right\|_{H^3}^2ds$$
$$+C_2\int_0^t\left\|(\sigma^{j+1}-\sigma^j,c^{j+1}-c^j)\right\|_{H^3}^2ds$$

$$\leq C(1+B_1{}^2)T_1 \sup_{0\leq t\leq T_1} \left\|(A^j(t)-A^{j-1}(t)\right\|_{H^3} + CB_1{}^2 \int_0^t \left\|\nabla(A^{j+1}(s)-A^j(s))\right\|_{H^3}^2 ds,$$

which by smallness of $B_1$ and $T_1$ implies that there is a constant $C_1 < 0$, there exists a constant $\theta\epsilon(0,1)$, such that for any $j \geq 1$

$$\sup_{0\leq t\leq T_1} \left\|(A^{j+1}(t)-A^j(t)\right\|_{H^3} \leq \theta \sup_{0\leq t\leq T_1} \left\|(A^j(t)-A^{j-1}(t)\right\|_{H^3},$$

which implies that $(A^j)_{J\geq 0}$ is a Cauchy sequence in the Banach space $C([0,T_1];H^3(\mathbb{R}^3))$.

By the property of Banach space, we have the limit function

$$A = A^0 + \lim_{i\to\infty}\sum_{j=0}^{i}\left(A^{j+1}-A^j\right)$$

exists in $C([0,T_1];H^3(\mathbb{R}^3))$, and satisfies

$$\sup_{0\leq t\leq T_1} \|A(t)\|_{H^3} \leq \sup_{0\leq t\leq T_1} \lim_{j\to\infty} inf \left\|A^j(t)\right\|_{H^3} \leq B_1. \qquad (3.21)$$

Finally, we show that the Cauchy problem (2.1)-(2.2) admits at most one solution in $C([0,T_1];H^3(\mathbb{R}^3))$. Suppose that there exist two solutions $A,\tilde{A}$ in $C([0,T_1];H^3(\mathbb{R}^3))$ which satisfy (3.6). Let $\tilde{\sigma}=\sigma_1(x,t)-\sigma_2(x,t)$, $\tilde{u}=u_1(x,t)-u_2(x,t)$, and $\tilde{c}=c_1(x,t)-c_2(x,t)$ solves

$$\partial_t \tilde{\sigma} + n_\infty\tilde{\sigma} - \Delta\tilde{\sigma} = u_1.\nabla\tilde{\sigma} - \tilde{u}.\nabla\sigma_2 + \nabla.(\sigma_2\nabla\tilde{c}) - \nabla.(\tilde{\sigma}\nabla c_1) - (\sigma_1+\sigma_2)\tilde{\sigma}$$

$$\partial_t \tilde{u} + \gamma\Delta\tilde{u} = -u_2\nabla\cdot\tilde{u} - \tilde{u}\nabla\cdot u_1 - \tilde{\sigma}\nabla\varphi$$

$$\partial_t \tilde{c} - \Delta\tilde{c} + n_\infty\tilde{c} = -u_2\nabla\cdot\tilde{c} - \tilde{u}.\nabla c_1 + \tilde{\sigma}c_2 + \sigma_1\tilde{c}. \qquad (3.22)$$

Multiplying $\tilde{\sigma}$ to both sides of the first equation of (3.22) and integrating over $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3}\tilde{\sigma}\,\partial_t\tilde{\sigma}dx + n_\infty\int_{\mathbb{R}^3}\tilde{\sigma}\,\tilde{\sigma}dx - \int_{\mathbb{R}^3}\tilde{\sigma}\,\Delta\tilde{\sigma}dx = \int_{\mathbb{R}^3}\tilde{\sigma}\,\nabla.(\tilde{\sigma}\nabla c_1)dx + \int_{\mathbb{R}^3}\tilde{\sigma}\,\nabla.(\sigma_2\nabla\tilde{c})dx + \int_{\mathbb{R}^3}\tilde{\sigma}\, u_1.\nabla\tilde{\sigma}dx - \int_{\mathbb{R}^3}\tilde{\sigma}\,\tilde{u}.\nabla\sigma_2 dx - \int_{\mathbb{R}^3}\tilde{\sigma}\,(\sigma_1+\sigma_2)\tilde{\sigma}dx.$$

Then, after using integration by parts and the Cauchy inequality, we obtain

$$\frac{d}{2\,dt}\|\tilde{\sigma}\|_{L^2}^2 + n_\infty\|\tilde{\sigma}\|_{L^2}^2 + \|\nabla\tilde{\sigma}\|_{L^2}^2$$
$$\leq c\|\nabla c_1\|_{L^\infty}\int_{\mathbb{R}^3}(|\tilde{\sigma}|^2+|\nabla\tilde{\sigma}|^2)dx$$
$$+ c\|\sigma_2\|_{L^\infty}\int_{\mathbb{R}^3}(|\nabla\tilde{\sigma}|^2+|\nabla\tilde{c}|^2)dx$$
$$+ c\|u_1\|_{L^\infty}\int_{\mathbb{R}^3}(|\tilde{\sigma}|^2+|\nabla\tilde{\sigma}|^2)dx + c\|\nabla\sigma_2\|_{L^\infty}\int_{\mathbb{R}^3}(|\tilde{\sigma}|^2+|\tilde{u}|^2)dx$$
$$+\|(\sigma_1+\sigma_2)\|_{L^\infty}\int_{\mathbb{R}^3}|\tilde{\sigma}|^2dx. \qquad (3.23)$$

For the estimate of $\tilde{u}$, multiplying $\tilde{u}$ to both sides of the second equation of (2.1) and taking integrations in x, we obtain

$$\int_{\mathbb{R}^3} \tilde{u}\, \partial_t \tilde{u} dx + \gamma \int_{\mathbb{R}^3} \tilde{u}\, \Delta \tilde{u} dx =$$

$$-\int_{\mathbb{R}^3} \tilde{u} \,.\, (\tilde{u} \nabla \cdot u_1) dx \; - \int_{\mathbb{R}^3} \tilde{u} \,.\, (u_2 \nabla \cdot \tilde{u}) dx + \int_{\mathbb{R}^3} \tilde{u} \,.\, (\tilde{\sigma} \nabla \varphi) dx.$$

By using integration by parts and the Cauchy inequality, we have

$$\frac{d}{2\,dt} \|\tilde{u}\|_{L^2}^2 + \gamma \|\nabla \tilde{u}\|_{L^2}^2 \le c \|\nabla \cdot u_1\|_{L^\infty} \|\tilde{u}\|_{L^2}^2 + \|u_2\|_{L^\infty} (\|\tilde{u}\|_{L^2}^2 + \|\nabla . \tilde{u}\|_{L^2}^2)$$

$$+ c \|\nabla \varphi\|_{L^\infty} \left( \|\tilde{u}\|_{L^2}^2 + \|\tilde{\sigma}\|_{L^2}^2 \right).$$

Since $L^\infty$ norms of $\sigma^i$, $u^i$, $c^i$ where i = 1, 2 are bounded, we have

$$\frac{d}{2\,dt} \|\tilde{u}\|_{L^2}^2 + c \|\nabla \tilde{u}\|_{L^2}^2 \le c \|\tilde{\sigma}\|_{L^2}^2 + c \|\tilde{u}\|_{L^2}^2. \qquad (3.24)$$

Similarly, as above, we estimate $\tilde{c}$ as follows:

$$\frac{d}{2\,dt} \|\tilde{c}\|_{L^2}^2 + C \|\nabla \tilde{c}\|_{L^2}^2 + n_\infty \|\tilde{c}\|_{L^2}^2 \le c\left( \|\tilde{c}\|_{L^2}^2 + \|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 \right). \qquad (3.25)$$

Then, after taking the linear combination of all estimates, we obtain

$$\frac{d}{2\,dt} \left( \|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2 \right) + \lambda_1 ((\|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2)$$

$$+ \lambda_2 ((\|\nabla \tilde{\sigma}\|_{L^2}^2 + \|\nabla \tilde{u}\|_{L^2}^2 + \|\nabla \tilde{c}\|_{L^2}^2) \le C\left( \|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2 \right). \qquad (3.26)$$

By applying Gronwall's inequality to the above equation, we have

$$\sup_{0 \le t \le T_1} \left( \|\tilde{\sigma}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2 \right) \le e^{cT_1} \left( \|\tilde{\sigma}(0)\|_{L^2}^2 + \|\tilde{u}(0)\|_{L^2}^2 + \|\tilde{c}(0)\|_{L^2}^2 \right).$$

Since the initial data of ($\tilde{\sigma}$, $\tilde{u}$, $\tilde{v}$,$\tilde{w}$) are all zero for T > 0, that implies the uniqueness of the local solution.

## CONCLUSION

In this paper, we prove the existence of local solutions for Navier Stokes system modeling cellular swimming in fluid drops in three dimensions. We show the existence of local solutions by the energy method. We divided the proof into four steps, using integral by parts, Cauchy –Schwarz inequality, and Gronwall's inequality to prove these steps.

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