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Homotopy Perturbation Method for Solving Mathematical Model of Brain Tumor Growth

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Abstract

The tumor growth models are vital and efficient tools for treating and diagnosing the disease. Therefore, we will find in this paper an approximate solution to the brain tumor growth model for a variable killing rate under medical treatment by applying the homotopy perturbation method (HPM). This method is both effective and simple, as it doesn't require the development of any iterative scheme to find a solution to the given equations. We will apply a new homotopy perturbation method (NHPM), which shortens the steps used in HPM by utilizing the first approximate solution to get the exact solution. The efficiency and reliability of the presented methods will be tested using some examples. Additionally, we will calculate the norm errors L_2, L_∞, and absolute error. Furthermore, we will conduct numerical simulations and generate graphics for this model using the Wolfram Mathematica 13.2 code.

Keywords: Brain tumor growth; Burgess equation; Homotopy perturbation method.

INTRODUCTION

Cancer cells grow and multiply very quickly, and most cancer treatments only rarely kill active-stage cells, this has prompted scientists to develop models of the growth of these tumors to develop effective treatment strategies and improve diagnostic and prevention methods. These models help improve patient care and also work to simulate the effects of different treatments on growth tumors and achieve the best results with the least side effects. In our study, we will study the growth model of a glioma brain tumor, which is commonly found in humans and can be managed with chemotherapy, radiotherapy, and surgery.in (Burgess et al., 1997) presented the initial formula for studying the glioma model, where they proposed a three-dimensional model for the growth of glioma that is devoid of any medical treatment and can grow without restrictions. This model was developed by many physicists, mathematicians, biologists, and medicines by incorporating cancer-killing substances into treatment. It was done by utilizing differential or integral equations, combining ideas derived from these sciences (Cruywagen et al., 1995; González-Gaxiola & Bernal-Jaquez, 2017; Wein & Koplow, 1999). In our research, we will study how to apply the homotopy perturbation method (HPM) and the modified homotopy perturbation method (NHPM) to a developed model of the Burgess equation, as this method was studied for the first time in (He, 1999) to solve nonlinear differential and integral equations. We will compare the approximate solution obtained from this method with the exact solution provided in the examples. This will be done using the absolute error and the norm errors L_2, L_∞. To calculate the results, we will use the Wolfram Mathematica 13.2 software.

Mathematical model of the Brain tumor growth (BTG):

Several scholars have discussed the Mathematical model describing brain tumor growth (Ganji et al., 2021; Tracqui et al., 1995).

The equation that expresses the tumor rate is (González-Gaxiola & Bernal-Jaquez, 2017; Nayied et al., 2023):

$$\frac{\partial n(x,\tau)}{\partial \tau} = D\nabla^2 n(x,\tau) + \rho(\tau)n(x,\tau)$$
$$= D\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial n(x,\tau)}{\partial x}\right) + \rho(\tau)n(x,\tau) \qquad (1)$$

Where $n(x,\tau)$ is the tumor cell concentration at the time t, $\nabla^2$is the Laplacian operator, D the diffusion coefficient, and $\rho(\tau)$ is the growth rate of the tumor. The equation (1) is known as the Burgess equation, but this model has been modified by adding the killing rate $k(\tau)$ to equation (1) by (Wein & Koplow, 1999) so the Burgess equation was obtained in the form:

$$\frac{\partial n(x,\tau)}{\partial \tau} = D\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial n(x,\tau)}{\partial x}\right) + \rho(\tau)n(x,\tau) - k(\tau)n(x,\tau) \qquad (2)$$

Equation (2) can be rewritten as

$$\begin{cases} \frac{\partial n(x,\tau)}{\partial \tau} = D\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial n(x,\tau)}{\partial x}\right) + \rho(\tau)n(x,\tau) - k(\tau)n(x,\tau) \\ n(x_0,\tau_0) = n_0 \end{cases} \qquad (3)$$

Following (Andriopoulos & Leach, 2006; Singha & Nahak, 2022), we propose the change of variables $t = 2D\tau, u(x,t) = x\, n(x,\tau), and\ w(x,t) = \frac{\rho-k}{2D}u(x,t)$ in equation (3), we get

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x} + w(x,t) \\ u(x,0) = f(x) \end{cases} \qquad (4)$$

Where $w(x,t)$ represents the source term.

MATERIAL AND METHODS

The iterative methods employed in this paper to find the approximate solution of the glioma brain tumor model will be introduced in this section (Kashkari & Saleh, 2017; Pal et al., 2023; Sobamowo, 2023).

Homotopy perturbation method (HPM)

Consider examining the nonlinear differential equation that follows

$$A(u) - f(r) = 0, r \in \delta \qquad (5)$$

With the conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma \qquad (6)$$

Where A, B, r, $f(r)$, Γ respectively, are a general differential operator, a boundary operator, a coordinate, a known function, and the boundary of the domain δ. Operator A can be split into two separate operators, L (linear operator) and N (nonlinear operator). Consequently, equation (5) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \qquad (7)$$

Using the homotopy technique, we can create a homotopy denoted as $v(r,p): \delta \times [0,1] \to \Re$. Which meets the following conditions:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \qquad p\epsilon[0,1],\ r\epsilon\delta \quad (8)$$

Where u_0 is an initial approximation of Eq. (5). Using the homotopy technique, we can assume that the solution of equation (8) is as follows:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots\cdots\cdots$$

By setting $p = 1$, we obtain the solution of the equation (5)

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots\cdots$$

New homotopy perturbation method

The idea of this method is similar to HPM. First, we consider the Eqs. (5), (6), (7). Using NHPM we construct the following homotopy:

$$H(v,p) = (1-p)[L(v) - u_0] + p[A(v) - f(r)] = 0 \qquad (9)$$

Where u_0, is as in Eq. (8) . By using the homotopy technique and assuming that the solution of equation (9) can be expressed as:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots\cdots\cdots \qquad (10)$$

By setting the initial approximation of Eq. (5) in the form

$$u_0 = \sum_{i=0}^{\infty} c_i(x) R_i(x)\ , R_i(x) = t^i \qquad (11)$$

Where, $c_i(x)$ are unknown coefficients and $R_i(x)$ are known functions. By substituting (10) and (11) into (9) and comparing the coefficients of p to the same powers, we assume $u_1(x,t) = 0$. Therefore the exact solution can be obtained as:

$$w(x,t) = f(x) + \sum_{i=0}^{\infty} c_i(x) \frac{R_{i+1}(x)}{i+1} \qquad (12)$$

Where $R_i(x) = t^i, c_i(x), i = 0,1,\cdots$ unidentified quantities that would be evaluated

Application of Iterative Methods in the Brain Tumor Growth Model

Application of Homotopy Perturbation Method in the Brain Tumor Growth Model

This part is dedicated to the analysis of the BTG model (4) by using HPM, where

$$L[u(x,t)] = u_t(x,t)\ , N[u(x,t)] = -\left(\frac{1}{2}u_{xx}(x,t) + w(x,t)\right), f(x,t) = 0$$

i.e. $A(u(x,t)) = u_t(x,t) - \frac{1}{2}u_{xx}(x,t) - w(x,t)$

Where $w(x,t) = \frac{\rho - k}{2D} u(x,t)$

By employing the homotopy technique, we obtain

$$H(u,p) = u_t - v_{0t} + p\left(v_{0t} - \frac{1}{2}u_{xx} - w(x,t)\right) \qquad (13)$$

Substituting the initial condition and $u = \sum_{i=0}^{\infty} p^i u_i$ in the above equation

In the first case, if $w(x,t) = au(x,t), a\ is\ constant$, then

$$\frac{\partial u_0}{\partial t} + p\frac{\partial u_1}{\partial t} + p^2\frac{\partial u_2}{\partial t} + \cdots - v_{0t} + p\left(v_{0t} - \frac{1}{2}\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \cdots\right) - a(u_0 + pu_1 + p^2 u_2 + \cdots)\right) = 0$$

By comparing the coefficient of terms with identical power of p, we get

$$p^0: u_{0t} - v_{0t} = 0 \Longrightarrow u_0 = u(x,0)$$

$$p^1: u_{1t} = au_0 + \frac{1}{2}u_{0xx} - v_{0t}\ , u_1(x,0) = 0$$

$$p^2: u_{2t} = au_1 + \frac{1}{2}u_{1xx}, u_2(x,0) = 0 \qquad (14)$$

$$p^3: u_{3t} = au_2 + \frac{1}{2}u_{2xx}, u_3(x,0) = 0$$

$$\vdots$$

$p^n: u_{nt} = au_{(n-1)} + \frac{1}{2}u_{(n-1)xx}, u_n(x,0) = 0$

By integrating both sides of the above equations for t, we get the required solution.

In the second case, if $w(x,t) = g\big(u(x,t)\big), i.e.\, w(x,t)$ is a nonlinear function of $u(x,t)$.

Using the same steps as in the first case, but setting $w(x,t) = g(u_0 + pu_1 + p^2u_2 + \cdots)$ and using Taylor's series in Eq. (13).

Application of New Homotopy Perturbation Methods in the Brain Tumor Growth Model

For solving Eq.4 by NHPM we construct the following homotopy:

$$u_t = v_0 - p\left(v_0 - \frac{1}{2}u_{xx} - w(x,t)\right) \tag{15}$$

Integrate both sides of the above equation for t

$$u(x,t) = u(x,0) + \int_0^t v_0 dt - p\int_0^t \left(v_0 - \frac{1}{2}u_{xx} - w(x,t)\right)dt \tag{16}$$

Where $u = \sum_{i=0}^{\infty} p^i u_i$, $v_0 = \sum_{i=0}^{\infty} c_i(x)R_i(t), R_i(t) = t^i$ in the eq. (16) and equating the coefficients of p with the same power leads to

In the first case, if $w(x,t) = au(x,t), a$ is a constant:

$$\begin{aligned}
p^0 &: u_0 = u(x,0) + \int_0^t (c_0 + c_1t + c_2t^2 + \cdots)dt \\
p^1 &: u_1 = -\int_0^t \left((c_0 + c_1t + c_2t^2 + \cdots) - \frac{1}{2}u_{0xx} - au_0\right)dt \\
p^2 &: u_2 = -\int_0^t \left(-\frac{1}{2}u_{1xx} - au_1\right)dt \\
p^3 &: u_3 = -\int_0^t \left(-\frac{1}{2}u_{2xx} - au_2\right)dt \\
&\vdots
\end{aligned} \tag{17}$$

And, so on

If we solve eq.'s (17) in a manner that

$$u_1 = -\int_0^t \left((c_0 + c_1t + c_2t^2 + \cdots) - \frac{1}{2}u_{0xx} - au_0\right)dt = 0$$

Then, the equations (17) get the yield to

$u_m = 0, m = 2,3,4,\cdots$

Thus, the exact solution can be obtained as follows:

$u(x,t) = f(x) + \sum_{i=0}^{\infty} c_i \frac{R_{i+1}}{i+1}, \quad R_i = t^i$

In the second case, if $w(x,t) = g\big(u(x,t)\big), i.e.\, w(x,t)$ is a nonlinear function of $u(x,t)$.

Using the same steps as in the first case, but setting $w(x,t) = g(u_0 + pu_1 + p^2u_2 + \cdots)$ and using Taylor's series in Eq. (16).

Numerical simulation

In this section, we will employ our methods to obtain an approximate solution of the BTG model (4) to verify the efficiency of the methods used in this paper. The given examples have been chosen from (González-Gaxiola & Bernal-Jaquez, 2017; Nayied et al., 2023).

The numerical simulation for the examples was performed using the Wolfram Mathematica code. To determine the quality of the results obtained, the following error norms L_2 and L_∞ are calculated

$$L_2 = \|u - u_{exact}\|_2 = \sqrt{\Delta x \Delta t \sum_{i=0}^{N}\sum_{j=0}^{M}\left(u(x_i,t_j) - u_{exact}(x_i,t_j)\right)^2}, \Delta x = \frac{1}{N-1}, \Delta t = \frac{1}{M-1}$$

$L_\infty = \|u-u_n\|_\infty = \max_{i,j}|u-u_{exact}|$, $abs.error = |u-u_{exact}|$

Example 4.1

We consider the following Burgess equation

$$u_t(x,t) = \frac{1}{2}u_{xx}(x,t) + w(x,t) \qquad (18)$$

With the initial condition

$$u(x,0) = e^x \qquad (19)$$

Where $w(x,t) = \frac{1}{2}u(x,t)$ and the exact solution $u(x,t) = e^{x+t}$

To find a solution of eq. (18) by HPM, we will follow the same steps used in part (3.1), we obtain the following:

$$p^0: u_0 = e^x$$
$$p^1: u_1 = te^x$$
$$p^2: u_2 = \frac{t^2}{2}e^x \qquad (20)$$
$$p^3: u_3 = \frac{t^3}{6}e^x$$
$$\vdots$$

Gives the series solution as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n = \left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)e^x$$
$$= e^{x+t}$$

Now, we will use NHPM to find the solution to equation (18) by following the same steps as in part (3.2).

$$p^0: u_0 = e^x + \int_0^t (c_0 + c_1 t + c_2 t^2 + \cdots)dt$$
$$p^1: u_1 = -\int_0^t \left((c_0 + c_1 t + c_2 t^2 + \cdots) - \frac{1}{2}u_{0xx} - \frac{1}{2}u_0\right)dt \qquad (21)$$
$$p^2: u_2 = -\int_0^t \left(-\frac{1}{2}u_{1xx} - \frac{1}{2}u_1\right)dt$$
$$p^3: u_3 = -\int_0^t \left(-\frac{1}{2}u_{2xx} - \frac{1}{2}u_2\right)dt$$
$$\vdots$$

And, so on

Now we will get the value of $u_1(x,t)$ such that the values $u_2, u_3, \ldots u_n$ will vanish.

$$-\int_0^t \left((c_0 + c_1 t + c_2 t^2 + \cdots) - \frac{1}{2}u_{0xx} - au_0\right)dt = 0 \qquad (22)$$

Now putting the coefficients of t equal to zero in eq.(22), which gives

$$c_0 = e^x, c_1 = \frac{1}{2}(c_0 + c_0^{"}) = e^x, c_2 = \frac{1}{4}(c_1 + c_1^{"}) = \frac{1}{4}e^x, c_3 = \frac{1}{6}(c_2 + c_2^{"}) = \frac{1}{6}e^x \text{ and so on}$$

$$c_n = \frac{1}{n!}e^x$$

So, the solution of eq. (18) is as follows:

$$u(x,t) = u_0(x,t) + \sum_{i=0}^{\infty} c_i \frac{R_{i+1}}{i+1}, R_i = t^i$$
$$= e^x + e^x t + e^x\frac{t^2}{2} + \frac{1}{2}e^x\frac{t^3}{3} + \frac{1}{6}e^x\frac{t^4}{4} + \cdots$$

$$= e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)$$

So, we get

$u(x,t) = e^{x+t}$

Which is an exact solution.

The numerical results of this example are displayed in Table 1 and Figures 1, 2, and 3. Table 1 compares the error norms $L_2,\ L_\infty$ for $N = 100$ at different time levels $t \leq 1$. Figure 1 displays the graphical behavior of the numerical solution at various time levels $t \leq 1$ and $i = 10$. Figure 2 compares HPM and NHM with the exact solution at $t = 0.5$. Moreover, Fig. 3 shows the absolute error between the solutions obtained by HPM and NHM and the exact solution for $x \in [0,1], t = 0.5, i = 10$. Based on Tables 1 and 2, as well as Figures 1, 2, and 3, it is clear that we obtained good results.

Table: (1). Comparison of the error norm $L_2,\ L_\infty$ at various times of ex.1

	Error	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
HPM		3.4542×10^{-16}	4.7468×10^{-16}	4.6524×10^{-16}	8.6455×10^{-16}	1.0117×10^{-15}
NHPM	L_2	3.4542×10^{-16}	4.7468×10^{-16}	4.6524×10^{-16}	8.6455×10^{-16}	1.0117×10^{-15}
HPM		$8,8818\times 10^{-16}$	1.3323×10^{-15}	8.8818×10^{-16}	2.6645×10^{-15}	3.5527×10^{-15}
NHPM	L_∞	$8,8818\times 10^{-16}$	1.3323×10^{-15}	8.8818×10^{-16}	2.6645×10^{-15}	3.5527×10^{-15}

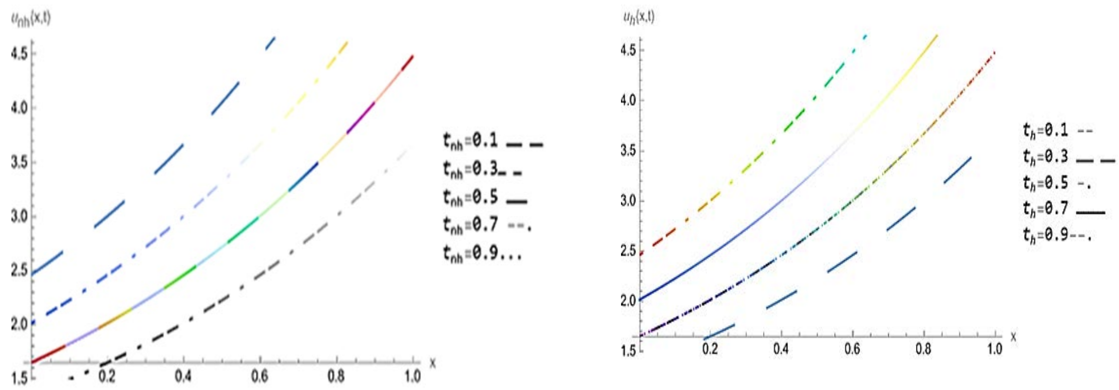

Figure: (1). Comparison of HPM versus NHM at different t of ex.1

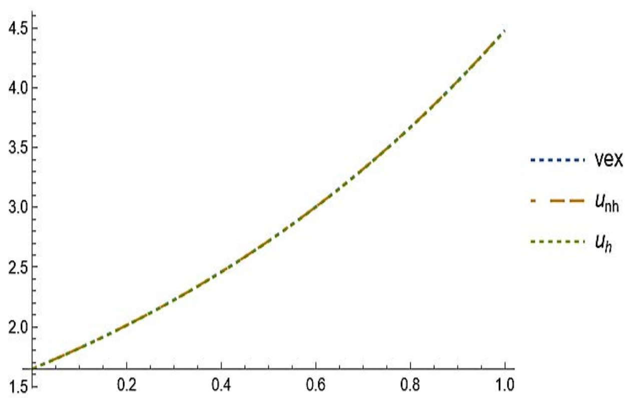

Figure: (2). Solution of HPM and NHM with exact solution at $x \in [0,1], t = 0.5$ of ex.1

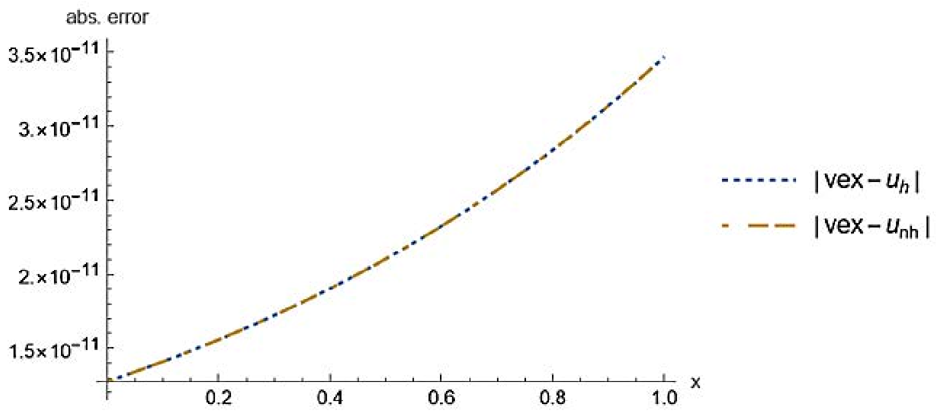

Figure: (3). Absolute error between solutions obtained by HPM and NHM for $x \in [0,1], t = 0.5, i = 10$ of ex.1

Example 4.2

Consider the nonlinear Burgess equation

$$u_t(x,t) = \frac{1}{2}u_{xx}(x,t) + w(x,t) \tag{23}$$

Subject to the initial condition

$$u(x,0) = \ln(x+2) \tag{24}$$

Where $w(x,t) = e^{-u(x,t)} + \frac{1}{2}e^{-2u(x,t)}$ and the exact solution $u(x,t) = \ln(x+t+2)$

In the first case, we will solve eq. 23 using the HPM by following the steps used in part (3.1) when $w(x,t)$ is a nonlinear function as follows:

$$\frac{\partial u_0}{\partial t} + p\frac{\partial u_1}{\partial t} + p^2\frac{\partial u_2}{\partial t} + \cdots - v_{0t} + p\left(v_{0t} - \frac{1}{2}\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \cdots\right) - e^{-(u_0+pu_1+p^2u_2+\cdots)} - \frac{1}{2}e^{-2(u_0+pu_1+p^2u_2+\cdots)}\right) = 0$$

By simplification of the above equation, we get

$$\frac{\partial u_0}{\partial t} + p\frac{\partial u_1}{\partial t} + p^2\frac{\partial u_2}{\partial t} + \cdots - v_{0t} + pv_{0t} - \frac{1}{2}p\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{2}p^2\frac{\partial^2 u_1}{\partial x^2} - \frac{1}{2}p^3\frac{\partial^2 u_2}{\partial x^2} - \cdots - pe^{-u_0}e^{-(pu_1+p^2u_2+\cdots)} - \frac{1}{2}pe^{-2u_0}e^{-2(pu_1+p^2u_2+\cdots)} = 0$$

By using the Taylor series to expand the previous expression, we obtain

$$\frac{\partial u_0}{\partial t} + p\frac{\partial u_1}{\partial t} + p^2\frac{\partial u_2}{\partial t} + \cdots - v_{0t} + pv_{0t} - \frac{1}{2}p\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{2}p^2\frac{\partial^2 u_1}{\partial x^2} - \frac{1}{2}p^3\frac{\partial^2 u_2}{\partial x^2} - \cdots$$

$$-p\left(e^{-u_0}\left(1 - (pu_1 + p^2u_2 + \cdots) + \frac{(pu_1 + p^2u_2 + \cdots)^2}{2!} - \cdots\right)\right) - \frac{1}{2}p\left(e^{-2u_0}\left(1 - 2(pu_1 + p^2u_2 + \cdots) + \frac{4(pu_1 + p^2u_2 + \cdots)^2}{2!} - \cdots\right)\right) = 0$$

Comparing the coefficients of equal powers of p

$p^0: u_0 = \ln(x+2)$

$$p^1: u_1 = \frac{t}{2+x} \tag{25}$$

$p^2: u_2 = -\frac{t^2}{2(2+x)^2}$

$p^3: u_3 = \frac{t^3}{3(2+x)^3}$

$\vdots$

Gives the solution as

$$u(x,t) = \ln(x+2) + \frac{t}{2+x} - \frac{t^2}{2(2+x)^2} + \frac{t^3}{3(2+x)^3} + \cdots$$

That gives the exact solution

$u(x,t) = \ln(x+t+2)$

In the second case, we will solve eq.23 using the NHPM. By following the steps in part (3.2), we get

$$u(x,t) = \ln(x+2) + \int_0^t v_0 dt - p\int_0^t \left(v_0 - \frac{1}{2}\left(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \cdots\right) - e^{-(u_0 - pu_1 + p^2 u_2 + \cdots)} - \frac{1}{2}e^{-2(u_0 - pu_1 + p^2 u_2 + \cdots)}\right)$$

Solving the above equation by using the homotopy technique, we get

$$p^0: u_0 = \ln(x+2) + \int_0^t v_0 dt$$

$$p^1: u_1 = -\int_0^t \left(v_0 - \frac{1}{2}u_{0xx} - e^{-u_0} - \frac{1}{2}e^{-2u_0}\right)dt \quad (26)$$

$$p^2: u_2 = -\int_0^t \left(-\frac{1}{2}u_{1xx} + u_1 e^{-u_0} + u_1 e^{-2u_0}\right)dt$$

$$p^3: u_3 = -\int_0^t \left(-\frac{1}{2}u_{2xx} + u_2 e^{-u_0} + u_2 e^{-2u_0} - \frac{1}{2}u_1^2 e^{-u_0} - u_1^2 e^{-2u_0}\right)dt$$

$\vdots$

By assuming $v_0 = \sum_{i=0}^{\infty} c_i(x)R_i(t), R_i(t) = t^i$, and solving the equation $u_1(x,t) = 0$, we get

$$c_0 = \frac{1}{2+x}, c_1 = -\frac{1}{(2+x)^2}, c_2 = \frac{1}{(2+x)^3}, c_3 = \frac{1}{(2+x)^4} \cdots, c_n = (-1)^n \frac{1}{(2+x)^{n+1}}$$

Moreover, we have

$$u(x,t) = u_0(x,t) + \sum_{i=0}^{\infty} c_i \frac{R_{i+1}}{i+1}, R_i = t^i$$

$$= \ln(x+2) + \frac{t}{2+x} - \frac{t^2}{2(2+x)^2} + \frac{t^3}{3(2+x)^3} - \frac{t^4}{4(2+x)^4} + \cdots$$

Thus,

$u(x,t) = \ln(x+t+2)$

Which is an exact solution.

Both the exact solution and the approximate solutions of ex.2 are compared in Fig. 4, table 3, and Table 4, where we notice that the solutions are almost identical, but through the absolute error, we notice that there is a small difference between the exact solution and the approximate solutions. Table 2 submits a comparison of the error norm L_2 for $(x,t) \in (0.1] \times [0,1], i = 10$ and different values of N, M.

Table: (2). The error norm L_2 on $(x,t) \in (0.1] \times [0,1]$of ex.2

	Error	$N = 10, M = 10$	$M = N = 100$	*M=4,N=8*
HPM	L_2	3.8586×10^{-6}	2.1628×10^{-6}	6.1596×10^{-6}
NHPM		1.7073×10^{6}	9.1124×10^{-7}	2.7615×10^{-6}
HPM	L_∞	3.046×10^{-5}	3.046×10^{-5}	3.046×10^{-5}
NHPM		1.3929×10^{-5}	1.3929×10^{-5}	1.3929×10^{-5}

Table: (3). Comparison between solutions obtained via HPM, NHM, and exact solution on $(x,t) \in (0.1] \times [0,1]$ of ex.2

	t=0.1			t=0.5			t=0.9		
x	PHM	NHM	exact	PHM	NHM	exact	PHM	NHM	exact
0	0.74194	0.74194	0.74194	0.91629	0.91629	0.91629	1.06470	1.06471	1.06471
0.1	0.78846	0.78846	0.78846	0.95551	0.95551	0.95551	1.09861	1.09861	1.09861
0.2	0.83291	0.83291	0.83291	0.99325	0.99325	0.99325	1.13140	1.13140	1.13140
0.3	0.87547	0.87547	0.87547	1.02962	1.02962	1.02962	1.16315	1.16315	1.16315
0.4	0.91629	0.91629	0.91629	1.06471	1.06471	1.06471	1.19392	1.19392	1.19392
0.5	0.95551	0.95551	0.95551	1.09861	1.09861	1.09861	1.22377	1.22378	1.22378
0.6	0.99325	0.99325	0.99325	1.13140	1.13140	1.13140	1.25276	1.25276	1.25276
0.7	1.02962	1.02962	1.02962	1.16315	1.16315	1.16315	1.28093	1.28093	1.28093
0.8	1.06471	1.06471	1.06471	1.19392	1.19392	1.19392	1.30833	1.30833	1.30833
0.9	1.09861	1.09861	1.09861	1.22378	1.22378	1.22378	1.33500	1.33500	1.33500
1	1.13140	1.13140	1.13140	1.25276	1.25276	1.25276	1.36098	1.36098	1.36098

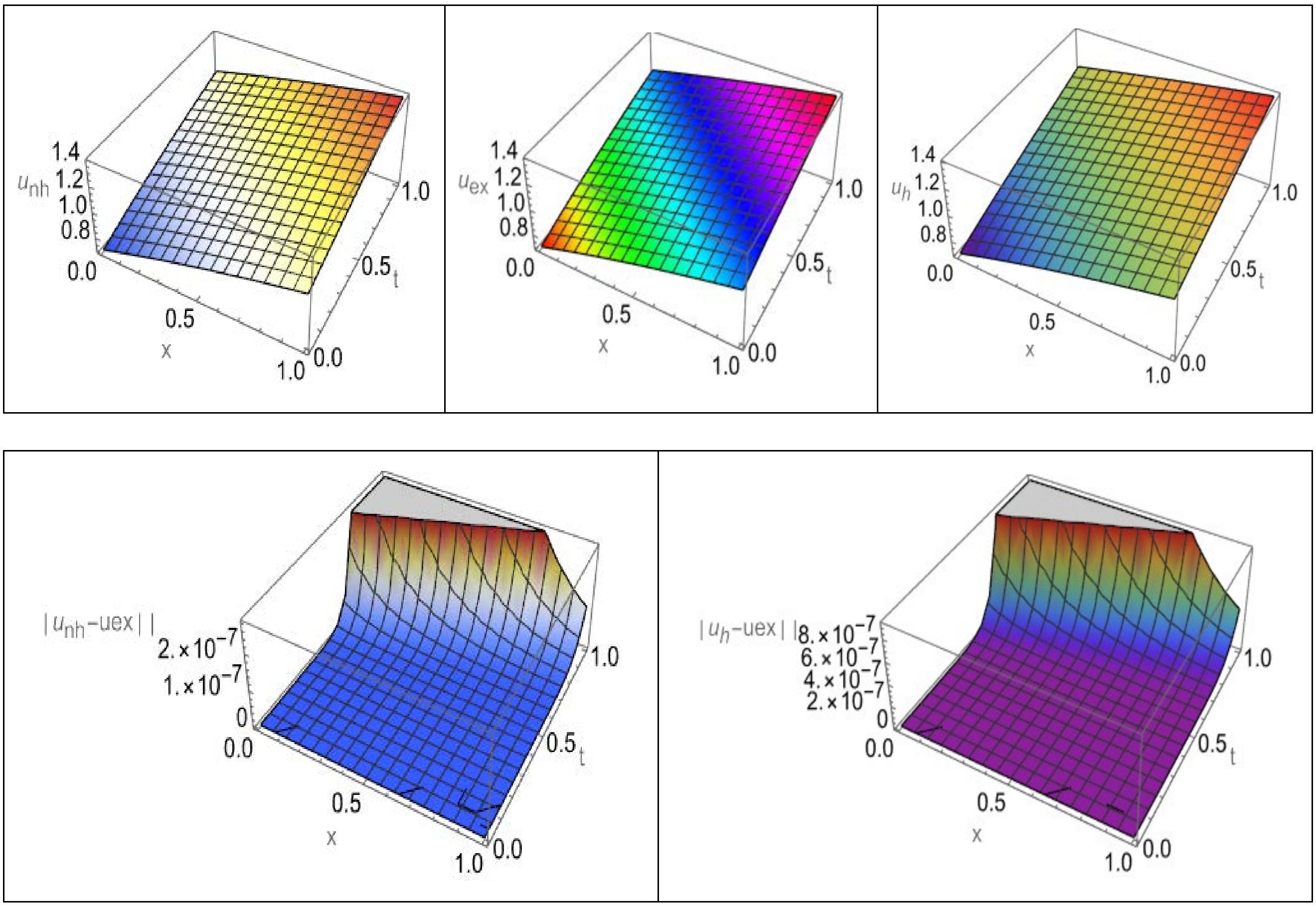

Figure: (4). Comparing between approximate solutions obtained via HPM, NHM, and exact solution on the interval $(x,t) \in (0.1] \times [0,1]$ of ex.2

Table: (4). The absolute error on $(x,t) \in (0.1] \times [0,1]$of ex.2

x	$t = 0.1$		$t = 0.5$		$t = 0.9$	
	$\lvert u_h - u_{ex} \rvert$	$\lvert u_{nh} - u_{ex} \rvert$	$\lvert u_h - u_{ex} \rvert$	$\lvert u_{nh} - u_{ex} \rvert$	$\lvert u_h - u_{ex} \rvert$	$\lvert u_{nh} - u_{ex} \rvert$
0	4.44×10^{-16}	1.11×10^{-16}	1.76×10^{-8}	4.03×10^{-9}	9.87×10^{-6}	4.06×10^{-6}
0.1	2.22×10^{-16}	1.11×10^{-16}	1.04×10^{-8}	2.26×10^{-9}	5.85×10^{-6}	2.29×10^{-6}
0.2	1.11×10^{-16}	0	6.29×10^{-9}	1.31×10^{-9}	3.55×10^{-6}	1.33×10^{-6}
0.3	3.33×10^{-16}	2.22×10^{-16}	3.89×10^{-9}	7.73×10^{-10}	2.20×10^{-6}	7.89×10^{-7}
0.4	2.22×10^{-16}	1.11×10^{-16}	2.45×10^{-9}	4.67×10^{-10}	1.40×10^{-6}	4.79×10^{-7}
0.5	0	1.11×10^{-16}	1.57×10^{-9}	2.88×10^{-10}	9.00×10^{-7}	2.96×10^{-7}
0.6	2.22×10^{-16}	0	1.03×10^{-9}	1.81×10^{-10}	5.90×10^{-7}	1.87×10^{-7}
0.7	0	2.22×10^{-16}	6.83×10^{-10}	1.16×10^{-10}	3.93×10^{-7}	1.20×10^{-7}
0.8	2.22×10^{-16}	2.22×10^{-16}	4.60×10^{-10}	1.52×10^{-11}	2.65×10^{-7}	7.82×10^{-8}
0.9	0	2.22×10^{-16}	3.14×10^{-10}	4.96×10^{-11}	1.82×10^{-7}	5.17×10^{-8}
1	2.22×10^{-16}	0	2.17×10^{-10}	3.32×10^{-11}	1.26×10^{-7}	3.47×10^{-8}

CONCLUSION

In this manuscript, we used the HPM and NHPM methods to solve the brain tumor growth model. After comparing the results obtained from various examples, we have concluded that the methods proposed in this study are effective and accurate for solving this mathematical model. We calculated the error norms L_2, L_∞ , and absolute error, and the results indicated that these error norms L_2, L_∞, and absolute errors are very small. Therefore, we can assert that the methods outlined in this paper yield good and reliable results. We utilized Wolfram Mathematica 13.2 software for performing numerical computations and generating 2D and 3D graphs relevant to this study.

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